Global instability in Hamiltonian systems and several Scattering maps

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First part Motivation: The model and the diffusion

The system

We consider the following *a priori unstable* Hamiltonian with $2+\frac{1}{2}$ degrees of freedom with 2π-periodic time dependence:

$$H_\varepsilon(p,q,I,\varphi,s) = \pm\left(\frac{p^2}{2}+\cos q - 1\right) + \frac{I^2}{2} + \varepsilon h(p,q,I,\varphi,s),$$

where p, $I \in \mathbb{R}$, q, φ, $s \in \mathbb{T}$, $h(p,q,I\varphi,s) = \cos q\,(a_{00} + a_{10}\cos\varphi + a_{01}\cos s)$ and ε is small enough.

Remark: for a C^2 generic h, the global instability was proved in (DLS06, DH09).

Defense of the model

- We want a simple model to give the zone in I of global instability with the following goals:
- To describe the map of heteroclinic orbits (Scattering map) and to design fast paths of instability.
- To estimate the time of diffusion.
- To play with parameter $\mu = a_{10}/a_{01}$ to prove global instability for all value of $\mu \neq 0, \infty$.
- To describe bifurcations of the scattering maps.

Possible limitation of the model Inner dynamics too simple: The restriction of the dynamics to the NHIM ($p = q = 0$) gives the integrable Hamiltonian $K(I, \varphi, s) = \frac{I^2}{2} + \varepsilon\,(a_{00} + a_{10}\cos\varphi + a_{01}\cos s)$. There are not "big gaps" between primary invariant tori.

We could choose other harmonics:

- $h(p, q, I, \varphi, s) = \cos q\,(a_{00} + a_{10}\cos(k\varphi + ls) + a_{01}\cos s)$, $k \neq 0$.
  In this case, the change $\varphi' = k\varphi + ls$ gives our model (with integrable Hamiltonian systems for the inner dynamics).
- $h(p, q, I, \varphi, s) = \cos q\,(a_{00} + a_{10}\cos(k\varphi + ls) + a_{01}\cos(k'\varphi + l's))$, with $\begin{vmatrix} k & s \\ k' & s' \end{vmatrix} \neq 0$.
  A change ($\varphi' = k\varphi + ls, s' = k'\varphi + l's$) gives the same scattering map. Inner dynamics is not integrable but is exponentially small in ε close to an integrable one.

In the unperturbed case, that is, $\varepsilon = 0$, the Hamiltonian H_0 is integrable (represents the standard pendulum plus a rotor):

$$H_0(p, q, I, \varphi, s) = \frac{p^2}{2} + \cos q - 1 + \frac{I^2}{2},$$

with associated equations:

$$\begin{aligned} \dot{q} &= \frac{\partial H_0}{\partial p} = p & \dot{p} &= -\frac{\partial H_0}{\partial q} = \sin q \\ \dot{\varphi} &= \frac{\partial H_0}{\partial I} = I & \dot{I} &= -\frac{\partial H_0}{\partial \varphi} = 0. \\ \dot{s} &= 1. \end{aligned} \tag{1}$$

and associated flow

$$\phi_t(p, q, I, \varphi, s) = (p(t), q(t), I, It + \varphi, t + s)\,.$$

I is constant.

Arnold diffusion

We have the following result:

Theorem

Consider a Hamiltonian of the form $H_\varepsilon(p,q,I,\varphi,t) = \frac{p^2}{2} + \cos q - 1 + \frac{I^2}{2} + \varepsilon\, f(q)\, g(\varphi,t)$, *where* $f(q) = \cos q$ *and* $g(\varphi,t) = a_{00} + a_{10}\, \cos\varphi + a_{01}\, \cos t$. *Assume that*

$$a_{10}\, a_{01} \neq 0$$

Then, for any $I^* > 0$, *there exists* $0 < \varepsilon^* = \varepsilon^*(I^*) << 1$ *such that for any* ε, $0 < \varepsilon < \varepsilon^*$, *there exists a trajectory* $(p(t), q(t), I(t), \varphi(t))$ *such that for some* $T > 0$

$$I(0) \leq -I^* < I^* \leq I(T).$$

We consider $\triangle I = \mathcal{O}(1)$, at least. This is an example of Arnold diffusion.

The dynamics associated to NHIM

We have two important dynamics associated to the system: the inner and the outer dynamics.

$$\widetilde{\Lambda} = \{\tau_I^0\}_{I\in[-I^*,I^*]} = \{(0,0,I,\varphi,s); I \in [-I^*,I^*], (\varphi,s) \in \mathbb{T}^2\}.$$

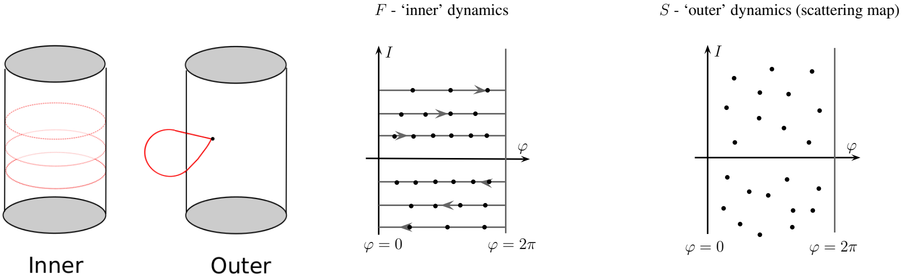
is a *Normally Hyperbolic Invariant Manifold* (NHIM), this set has stable and unstable invariant manifolds.

- The *inner* is the dynamics restricted to $\widetilde{\Lambda}$. (Inner map)
- The *outer* is the dynamics restricted to its invariant manifolds. (Scattering map)

Remark: In our case $\widetilde{\Lambda} = \widetilde{\Lambda}_\varepsilon$.

Inner and outer dynamics

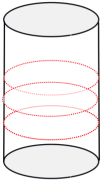
The unperturbed case, $\varepsilon = 0$



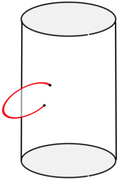
- Stable and unstable manifolds are *coincident.*
- The outer dynamics is the identity.

Inner and outer dynamics

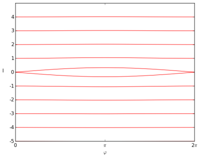
The perturbed case, $\varepsilon \neq 0$:



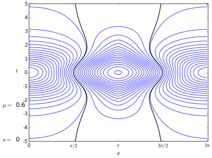
Inner



Outer



(a) Inner



(b) Outer

- Stable and unstable manifolds, in general, are not coincident.
- The outer dynamics ensures the growth of I, that is, the Arnold diffusion.

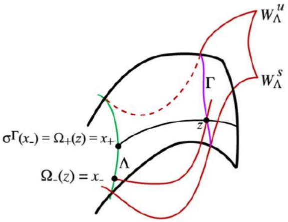
Outer dynamics: Scattering maps

Let $\widetilde{\Lambda}$ be a NHIM with invariant manifolds intersecting transversally along a homoclinic manifold Γ. A scattering map is a map S defined by $S(\tilde{x}_-) = \tilde{x}_+$ if there exists $\tilde{z} \in \Gamma$ satisfying

$$|\phi_t^\varepsilon(\tilde{z}) - \phi_t^\varepsilon(\tilde{x}_-)| \longrightarrow 0 \text{ as } t \longrightarrow -\infty$$
$$|\phi_t^\varepsilon(\tilde{z}) - \phi_t^\varepsilon(\tilde{x}_+)| \longrightarrow 0 \text{ as } t \longrightarrow +\infty,$$

that is, $W_\varepsilon^u(\tilde{x}_-)$ intersects transversally $W_\varepsilon^s(\tilde{x}_+)$ in $\tilde{z}$.

- S is locally well defined.



The Scattering map : Equations

$S(I, \varphi, s)$ is symplectic and exact (Delshams -de la Llave - Seara 2008), this implies that S takes the form:

$$S_\varepsilon(I, \varphi, s) = \left(I + \varepsilon \frac{\partial L^*}{\partial \varphi}(I, \varphi, s) + \mathcal{O}(\varepsilon^2), \varphi - \varepsilon \frac{\partial L^*}{\partial I}(I, \varphi, s) + \mathcal{O}(\varepsilon^2), s \right),$$

or simply

$$\mathcal{S}_\varepsilon(I, \theta) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I, \theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I, \theta) + \mathcal{O}(\varepsilon^2) \right),$$

where $\theta = \varphi - Is$ and $\mathcal{L}^*(I, \theta)$ is the Reduced Poincaré function.

So, our focus will be the level curves of $\mathcal{L}^*(I, \theta)$.

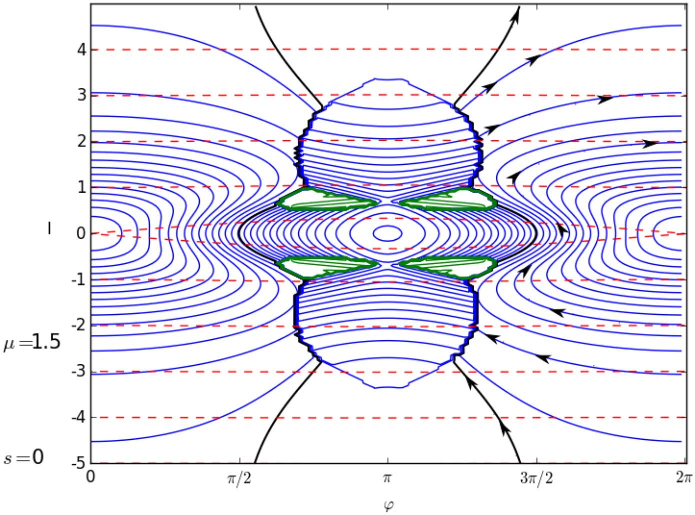
Remark: The variable s remains fixed under the action of the Scattering map, or plays the role of a parameter.

Pseudo-orbits : ways of diffusion

Basically, we ensure the Arnold diffusion performing the following scheme:

- To construct a composition of some Scattering map and some Inner map. This composition is called a *pseudo-orbit*.
- To use previous results about Shadowing (Gidea-de la Llave - Seara 2014) for ensuring the existence of a real orbit close to our pseudo-orbit.

A first example of pseudo-orbit



Special Pseudo orbits: Highways

Recall:

- Our perturbation is $\varepsilon \cos q \,(a_{00} + a_{10} \cos \varphi + a_{01} \cos s)$.
- the only hypothesis about it is $a_{10} a_{01} \neq 0$.

We have special curves that we called Highways. In concrete:

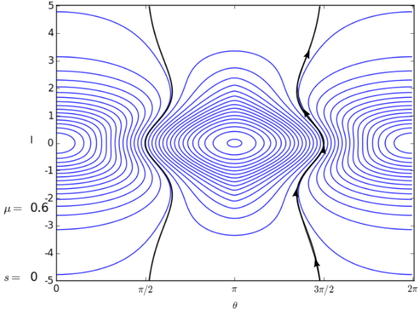
Definition: Highways

Highways are the level curves of $\mathcal{L}^*$ such that

$$\mathcal{L}^*(I, \theta) = 4a_{00} + \frac{2\pi a_{01}}{\sinh(\pi/2)}.$$

Properties of highways

- Highways are "vertical"
- We always have a "pair" of highways. One goes up, the other goes down (this depends on signal of a_{10}/a_{01}.)
- It is easy to construct pseudo-orbits where highways are defined.



Second part Structure and properties of Scattering map

Melnikov Potential

Note that for scattering maps we have to look for homoclinic points of $\tilde{\Lambda}$. We will use the Melnikov Potential:

Proposition

Given $(I, \varphi, s) \in [-I^*, I^*] \times \mathbb{T}^2$, assume that the real function

$$\tau \in \mathbb{R} \longmapsto \mathcal{L}(I, \varphi - I\,\tau, s - \tau) \in \mathbb{R}$$

has a non degenerate critical point $\tau^* = \tau(I, \varphi, s)$, where $\mathcal{L}(I, \varphi, s) =$

$$\int_{-\infty}^{+\infty} h(p_0(\sigma), q_0(\sigma), I, \varphi + I\sigma, s + \sigma; 0) - h(0, 0, I, \varphi + I\sigma, s + \sigma; 0) d\sigma.$$

Then, for $0 < |\varepsilon|$ small enough, there exists a transversal homoclinic point $\tilde{z}$ to $\widetilde{\Lambda}_\varepsilon$, which is ε-close to the point
$\tilde{z}^*(I, \varphi, s) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) \in W^0(\widetilde{\Lambda})$:

$$\tilde{z} = \tilde{z}(I, \varphi, s) = (p_0(\tau^*) + O(\varepsilon), q_0(\tau^*) + O(\varepsilon), I, \varphi, s) \in W^u(\widetilde{\Lambda}_\varepsilon) \pitchfork W^s(\widetilde{\Lambda}_\varepsilon).$$

Melnikov Potential and Reduced Poincaré function

- $\mathcal{L}$ is the Melnikov potential.
- In our model, $h(p,q,I,\varphi,s) = \cos q\,(a_{00} + a_{01}\cos\varphi + a_{01}\cos s)$.
- In our case

$$\mathcal{L}(I,\varphi,s) = A_{00} + A_{10}(I)\cos\varphi + A_{01}\cos s,$$

where $A_{00} = 4\,a_{00}$, $A_{10}(I) = \dfrac{2\,\pi\,I\,a_{10}}{\sinh(\frac{I\,\pi}{2})}$ and $A_{01} = \dfrac{2\,\pi\,a_{01}}{\sinh(\frac{\pi}{2})}$.

Remark: h, $\mathcal{L}$ are trigonometric polynomials of degree one in (φ, s).

Definition

Reduced Poincaré function is

$$\mathcal{L}^*(I,\theta) = \mathcal{L}(I, \varphi - I\,\tau^*(I,\varphi,s), s - \tau^*(I,\varphi,s)),$$

where $\theta = \varphi - I\,s$.

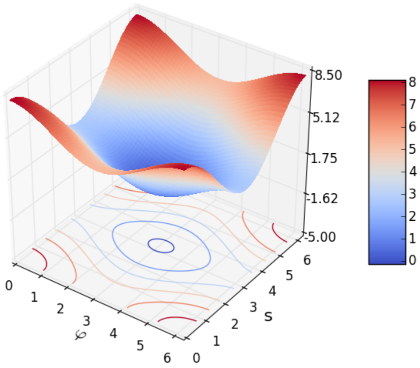


Figure : The Melnikov Potential, $\mu = 0.6$ and $I = 1$.

Intersection point between invariant manifolds:

We look for τ^* such that $\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\,\tau^*, s - \tau^*) = 0$. In our case, we look for τ^* such that:

$$I\,A_{10}(I)\sin(\varphi - I\,\tau^*) + A_{10}\sin(s - \tau^*) = 0. \tag{2}$$

Different view-points of $\tau^*(I, \varphi, s)$

- Critical points of $\mathcal{L}$ on the straight line $R(I, \varphi, s) = \{(\varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$.
- Intersection between $R(I, \varphi, s) = \{(\varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$ and the crest which it is the curve of equation

$$I\,A_{10}(I)\sin\varphi + A_{01}\sin s = 0.$$

.

Crests

Definition - Crests (Delshams-Huguet 2011)

For each I, we call *crests* the pair (φ, s) such that $\tau^* = 0$ satisfies the equation (2), that is,

$$I\, A_{10}(I) \sin\varphi + A_{01} \sin s = 0. \tag{3}$$

For the computation of the reduced Poincaré function, we have to study this equation. We can rewrite it as

$$\mu\alpha(I)\, \sin\varphi + \sin s = 0, \tag{4}$$

where

$$\alpha(I) = \frac{\sinh(\frac{\pi}{2})\, I^2}{\sinh(\frac{\pi\, I}{2})} \tag{5}$$

and

$$\mu = \frac{a_{10}}{a_{01}}. \tag{6}$$

Geometrical interpretation of the crest

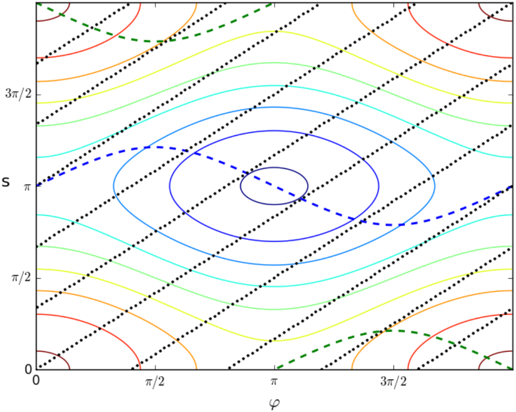


Figure : Level curves of $\mathcal{L}$ for $\mu = a_{10}/a_{01} = 0.5$ and $I = 1.2$.

- $(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π) always belong to the crest. One maximum and one minimum point and two saddle points.
- $\mathcal{L}^*(I,\theta)$ is $\mathcal{L}$ evaluated on the crest.
- $\theta = \varphi - Is$ is constant on the straight line $R(I,\varphi,s)$

Geometrical interpretation of the crest

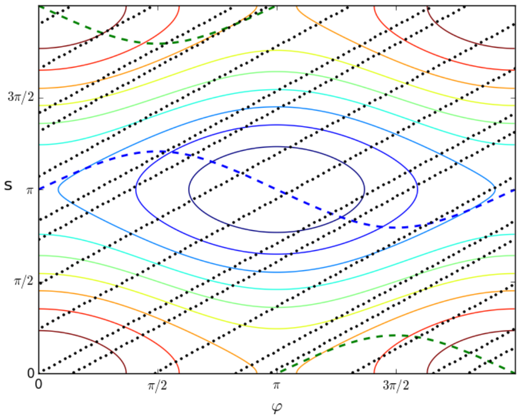


Figure : Level curves of $\mathcal{L}$ for $\mu = a_{10}/a_{01} = 0.5$ and $I = \sqrt{2}$.

Since $(\varphi, s) \in \mathbb{T}^2$,

- $R(I, \varphi, s)$ is a closed line if $I \in \mathbb{Q} \Rightarrow R(I, \varphi, s)$ intersects each crest $C(I)$, at most, on a finite number of points.
- $R(I, \varphi, s)$ is a dense line on $\mathbb{T}^2$ if $I \notin \mathbb{Q} \Rightarrow R(I, \varphi, s)$ intersects each crest $C(I)$ on an infinite number of points.

$\mathcal{L}^*$ is well defined if we restricted the domain:

Our restriction: Consider only the first intersection point, that is the homoclinic primary points.

Understanding the behavior of the crests

$\Downarrow$

Understanding the behavior of the Reduced Poincaré function

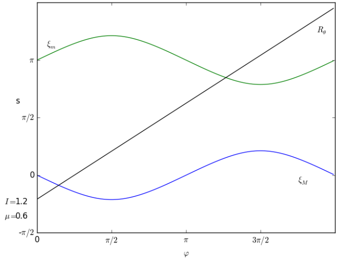
$\Downarrow$

Understanding the Scattering map

$0 < |\mu| < 0.97$

- $|\mu\alpha(I)| < 1$, there are two crests $\mathcal{C}_{\mathsf{M,m}}(I)$ parameterized by:

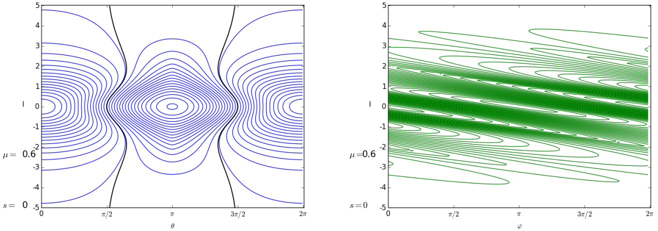
$$
\begin{aligned}
s = \xi_M(I,\varphi) &= -\arcsin(\alpha(I,\mu)\sin\varphi) \quad \mod 2\pi \qquad (7)\\
\xi_m(I,\varphi) &= \arcsin(\alpha(I,\mu)\sin\varphi) + \pi \quad \mod 2\pi
\end{aligned}
$$



They are the horizontal crests

$0 < |\mu| < 0.625$

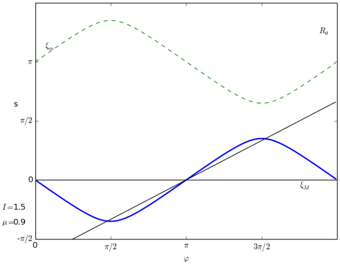
- For each I, the line $R(I, \varphi, s)$ and the crest $\mathcal{C}_{\mathrm{M,m}}(I)$ have only one intersection point.
- The intersection is always transversal.
- We have well defined S_M and S_m, where S_{M} is the scattering map associated to the intersections between $\mathcal{C}_{\mathrm{M}}(I)$ and $R(I, \varphi, s)$ and S_{m} is the scattering map associated to the intersection between $\mathcal{C}_{\mathrm{m}}(I)$ and $R(I, \varphi, s)$.



(a) Level curve of $\mathcal{L}_M^*(I, \theta)$. (b) Level curves of $\mathcal{L}_m^*(I, \theta)$

$0.625 < |\mu|$

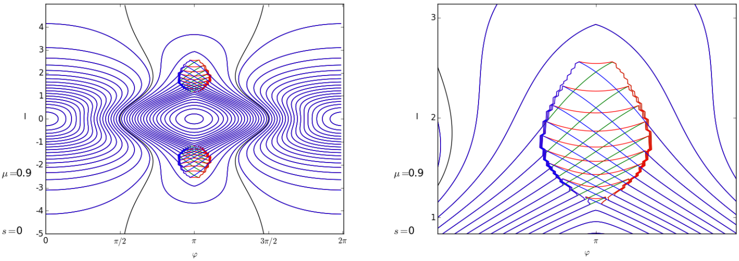
- The equations of the crests are the same.
- There are tangencies between $\mathcal{C}_{\mathrm{M,m}}(I,\varphi)$ and $R(I,\varphi,s)$. If $\theta \neq \pi$, the tangency happens for two angles. In this case, for some value of (φ, s), there are 3 points in $R(I,\varphi,s) \cap \mathcal{C}_{\mathrm{M,m}}(I)$.
- The item above implies that there are three scattering maps associated to each crest. In this case we have Multiple Scattering maps with different domains.



We define as tangency locus the set

$$\left\{(I,\theta);\frac{\partial\xi}{\partial\varphi}(I,\varphi)=\frac{1}{I}\right\}.$$

- Out of the delimited region by the tangency locus: Scattering maps are equal.
- In this region, they are different.



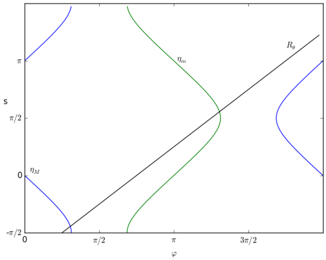
(c) The three types of level curves. (d) Zoom around the tangency locus

Figure : Level curves of $\mathcal{L}^*_M(I,\theta)$, $\mathcal{L}^{*(1)}_M(I,\theta)$ and $\mathcal{L}^{*(2)}_M(I,\theta)$

$|\mu| > 0.97$

- For some values of I, $|\mu\alpha(I)| > 1$, the two crests $\mathcal{C}_{\mathrm{M,m}}$ are parameterized by:

$$\begin{aligned} \varphi = \eta_M(I,s) &= -\arcsin(\alpha(I,\mu)\sin s) \mod 2\pi \\ \eta_m(I,s) &= \arcsin(\alpha(I,\mu)\sin s) + \pi \mod 2\pi \end{aligned} \tag{8}$$



They are the vertical crests

As this happens for some values of I and when it happens, we can look this crests locally as the horizontal crests, we restrict the domain of the Scattering map.

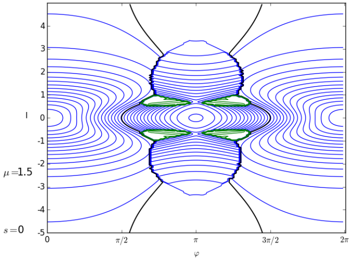


Figure : The level curves of $\mathcal{L}^*_{\mathsf{M}}(I,\theta)$, $\mu = 1.5$.

In green, the region where the scattering map S_{M} is not defined.

Several Scattering maps of several values of s

In this talk we have just displayed Scattering maps with $s = 0$. But if we change its value in the formula

$$S_\varepsilon(I,\varphi,s) = \left(I + \varepsilon \frac{\partial L^*}{\partial \varphi}(I,\varphi,s) + \mathcal{O}(\varepsilon^2), \varphi - \varepsilon \frac{\partial L^*}{\partial I}(I,\varphi,s) + \mathcal{O}(\varepsilon^2), s\right),$$

we have more options for the diffusion, that is, the pseudo-orbit.

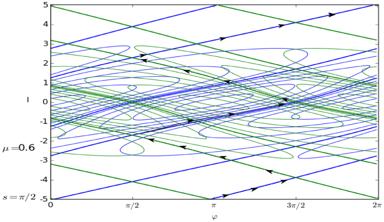


Figure : The level curves of the Reduced Poincaré function associated to $\mathcal{C}_\mathsf{M}(I)$ in blue, and associated to $\mathcal{C}_\mathsf{m}(I)$ in green, $s = \pi/2$.

Third part Diffusion using only highways and Time of diffusion

Domain of the Highways

Proposition

Consider the function $\mathcal{L}^*_M(I,\theta) = A_{00} + A_{10}(I)\cos(\theta - I\tau^*_M(I,\theta)) + A_{01}\cos(-\tau^*_M(I,\theta))$. *Assume that* $a_{10}\, a_{01} \neq 0$. *Then the highways is a union of two "vertical" curves on the axis* $\theta \times I$, *when* I *is in a set* B, *where*

- $|\mu| < 0.625$:$B = [0, +\infty)$
- $0.625 \leq |\mu|$:$B = [0, I_+) \cup (I_{++}, +\infty)$, *where*

$$I_{++} = \max\left\{ I > 0 : \frac{I^3 \sinh(\pi/2)}{\sinh(I\pi/2)} = \frac{1}{|\mu|} \right\}.$$

and

- $|\mu| \leq 1$:

$$I_+ = \min\left\{ I > 0 : \frac{I^3 \sinh(\pi/2)}{\sinh(I\pi/2)} = \frac{1}{|\mu|} \right\}$$

- $|\mu| \geq 1$:

$$I_+ = \min\left\{ I > 0 : \frac{I^2 \sinh(\pi/2)}{\sinh(I\pi/2)} = \frac{1}{|\mu|} \right\}$$

Diffusion on Highways

Theorem

Consider a Hamiltonian of the form
$H_\varepsilon(p,q,I,\varphi,t) = \frac{p^2}{2} + \cos q - 1 + \frac{I^2}{2} + \varepsilon\, f(q)\, g(\varphi,t)$, *where* $f(q) = \cos q$ *and* $g(\varphi,t) = a_{00} + a_{10}\ \cos\varphi + a_{01}\ \cos t$. *Assume that*

$$a_{10}\, a_{01}\ \neq\ 0$$

Then, for any I^* *there exists* $\varepsilon^* = \varepsilon^*(I^*) > 0$ *and* $0 < \varepsilon < \varepsilon^*$, *there exists a trajectory* $(p(t), q(t), I(t), \varphi(t))$ *such that for some* $T > 0$

$$I(0) \leq -I^*; \qquad I(T) \geq I^*,$$

where

- $|\mu| < 0.625$, I^* *is any* $I \in (0, +\infty)$.
- $0.625 \leq |\mu| \leq 1$, $I^* \in (0, I_+)$, *where* $I_+ = \min\{I > 0 : I^3 \sinh(\pi/2)/\sinh(\pi I/2) = 1/|\mu|\}$.
- $|\mu| \geq 1$, $I^* \in (0, I_+)$, *where* $I_+ = \{I > 0 : I^2 \sinh(\pi/2)/\sinh(\pi I/2) = 1/|\mu|\}$.

Time of diffusion

An estimate of the total time of diffusion between I_0 and I_{f}, for simplicity only along the highways is

$$T_d \sim N_{\mathsf{s}} T_{\mathsf{h}} \sim \frac{T_{\mathsf{s}}}{\varepsilon} \log\left(\frac{C_{\mathsf{h}}}{\varepsilon}\right),$$

where

- $T_{\mathsf{h}} \approx \log\left(\frac{C_{\mathsf{h}}}{\varepsilon}\right)$ is the time along the homoclinic invariant manifold of $\widetilde{\Lambda}$, where $C_{\mathsf{h}} = 8\,|a_{10}|\left(1 + \dfrac{1.465}{\sqrt{1-\mu^2\alpha^2(I_{\mathsf{M}})}}\right)$
- $N_{\mathsf{s}} = T_{\mathsf{s}}/\varepsilon$ is the number of iterates of the scattering map along the highway and
- $T_s = \displaystyle\int_{I_0}^{I_f} \frac{-\sinh(I\pi/2)}{2\pi I a_{10} \sin\psi_{\mathsf{h}}(I)} dI$, where $\psi_{\mathsf{h}} = \theta - I\tau^*(I,\theta)$ is a parametrization of the highway.

This estimate agrees with the optimal estimate of (Berti-Biasco-Bolle 2003) and (Treschev 2004), that is, a time of the order $\mathcal{O}(\varepsilon^{-1}\log\varepsilon^{-1})$.

Moltes gràcies!

A short bibliography

- A geometric mechanism of diffusion: Rigorous verification in a priori unstable Hamiltonian systems - (Delshams - Huguet 2011)
- A Geometric Approach to the Existence of Orbits with Unbounded Energy in Generic Periodic Perturbations by a Potential of Generic Geodesic Flows of $\mathbb{T}^2$ (Delshams - de la Llave -Seara 2000) (for Scattering maps)
- A general mechanism of diffusion in Hamiltonian systems: Qualitative results (Gidea - de la Llave - Seara 2014) (for Shadowing)
- Drift in phase space: a new variational mechanism with optimal diffusion time (Berti - Biasco - Bolle 2003)
- Evolution of slow variables in a priori unstable Hamiltonian systems (Treschev - 2004)