Global instability in Hamiltonian systems

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The system

We consider the following *a priori unstable* Hamiltonian with $2 + \frac{1}{2}$ degrees of freedom with 2π-periodic time dependence:

$$H_\varepsilon(p, q, I, \varphi, s) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2} + \varepsilon h(p, q, I, \varphi, s), \quad (1)$$

where p, $I \in \mathbb{R}$, q, φ, $s \in \mathbb{T}$, ε small enough and

$$h(p, q, I\varphi, s) = \cos q \left(a_0 \cos(k_1\varphi + l_1 s) + a_1 \cos(k_2\varphi + l_2 s)\right), \quad (2)$$

where $h(p, q, I, \varphi, s)$ is a perturbation which depends on two harmonics ($k_1 l_2 \neq k_2 l_1$ and $k_1 l_2 \neq 0$).

Goals

- To describe the maps of heteroclinic orbits (Scattering maps) and to design paths of instability.
- To estimate the time of diffusion (at least for $k_1 = l_2 = 1$ and $l_1 = k_2 = 0$).
- To play with the parameter $\mu = a_0/a_1$ to prove global instability for all value of $\mu \neq 0, \infty$.
- To describe bifurcations of the scattering maps.

In the unperturbed case, that is, $\varepsilon = 0$, the Hamiltonian H_0 is integrable (represents the standard pendulum plus a rotor) and takes the form

$$H_0(p, q, I, \varphi, s) = \frac{p^2}{2} + \cos q - 1 + \frac{I^2}{2}.$$

I is constant.

Arnold diffusion

For $\varepsilon \neq 0$, we have the following result

Theorem

Consider a Hamiltonian $H_\varepsilon(p, q, I, \varphi, t)$ *of the form* (1)*, where* $h(q, \varphi, s)$ *is given by* (2)*. Assume that* $a_0\, a_1 \neq 0$*.*
Then, for any $I^* > 0$*, there exists* $0 < \varepsilon^* = \varepsilon^*(I^*) << 1$ *such that for any* ε*,* $0 < \varepsilon < \varepsilon^*$*, there exists a trajectory* $(p(t), q(t), I(t), \varphi(t))$ *such that for some* $T > 0$

$$I(0) \leq -I^* < I^* \leq I(T).$$

We consider $\triangle I = \mathcal{O}(1)$, at least. This is an example of Arnold diffusion.

Pseudo-orbits : ways of diffusion

Basically, we ensure the Arnold diffusion performing the following scheme:

- To construct a composition of some Scattering map and some Inner map. This composition is called a *pseudo-orbit*.
- To use previous results about Shadowing (Gidea - de la Llave - Seara 2014) for ensuring the existence of a real orbit close to our pseudo-orbit.

The dynamics associated to NHIM

We have two important dynamics associated to the system: the inner and the outer dynamics.

$$\widetilde{\Lambda} = \{\tau_I^0\}_{I\in[-I^*,I^*]} = \{(0,0,I,\varphi,s); I \in [-I^*,I^*], (\varphi,s) \in \mathbb{T}^2\}.$$

is a *Normally Hyperbolic Invariant Manifold* (NHIM), this set has stable and unstable invariant manifolds.

- The *inner* is the dynamics restricted to $\widetilde{\Lambda}$. (Inner map)
- The *outer* is the dynamics restricted to its invariant manifolds. (Scattering map)

Remark: In our case $\widetilde{\Lambda} = \widetilde{\Lambda}_\varepsilon$.

Outer dynamics: Scattering maps

Let $\widetilde{\Lambda}$ be a NHIM with invariant manifolds intersecting transversally along a homoclinic manifold Γ. A scattering map is a map S defined by $S(\tilde{x}_-) = \tilde{x}_+$ if there exists $\tilde{z} \in \Gamma$ satisfying

$$|\phi_t^\varepsilon(\tilde{z}) - \phi_t^\varepsilon(\tilde{x}_\mp)| \longrightarrow 0 \text{ as } t \longrightarrow \mp\infty$$

that is, $W_\varepsilon^u(\tilde{x}_-)$ intersects transversally $W_\varepsilon^s(\tilde{x}_+)$ in $\tilde{z}$. S is symplectic and exact (Delshams -de la Llave - Seara 2008), this implies that S takes the form:

$$\mathcal{S}_\varepsilon(I,\theta) = \left(I + \varepsilon\,\frac{\partial \mathcal{L}^*}{\partial \theta}(I,\theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon\,\frac{\partial \mathcal{L}^*}{\partial I}(I,\theta) + \mathcal{O}(\varepsilon^2) \right),$$

where $\theta = \varphi - Is$ and $\mathcal{L}^*(I,\theta)$ is the Reduced Poincaré function.

So, our focus will be the level curves of $\mathcal{L}^*(I,\theta)$.

Remark: The variable s remains fixed under the action of the Scattering map, or plays the role of a parameter.

Melnikov Potential

Note that for scattering maps we have to look for homoclinic points of $\tilde{\Lambda}$. We will use the Melnikov Potential:

Proposition

Given $(I, \varphi, s) \in [-I^*, I^*] \times \mathbb{T}^2$, assume that the real function

$$\tau \in \mathbb{R} \longmapsto \mathcal{L}(I, \varphi - I\,\tau, s - \tau) \in \mathbb{R}$$

has a non degenerate critical point $\tau^* = \tau(I, \varphi, s)$, where $\mathcal{L}(I, \varphi, s) =$

$$\int_{-\infty}^{+\infty} h(p_0(\sigma), q_0(\sigma), I, \varphi + I\sigma, s + \sigma; 0) - h(0, 0, I, \varphi + I\sigma, s + \sigma; 0) d\sigma.$$

Then, for $0 < |\varepsilon|$ small enough, there exists a transversal homoclinic point $\tilde{z}$ to $\widetilde{\Lambda}_\varepsilon$, which is ε-close to the point $\tilde{z}^*(I, \varphi, s) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) \in W^0(\widetilde{\Lambda})$:

$$\tilde{z} = \tilde{z}(I, \varphi, s) = (p_0(\tau^*) + O(\varepsilon), q_0(\tau^*) + O(\varepsilon), I, \varphi, s) \in W^u(\widetilde{\Lambda}_\varepsilon) \pitchfork W^s(\widetilde{\Lambda}_\varepsilon).$$

Melnikov Potential and Reduced Poincaré function

- $\mathcal{L}$ is the Melnikov potential.
- In our model,
  $h(p,q,I,\varphi,s) = \cos q\,(a_0 \cos(k_1\varphi + l_1 s) + a_1 \cos(k_2\varphi + l_2 s))$.
- In our case

$$\mathcal{L}(I,\varphi,s) = A_0(I)\cos(k_1\varphi + l_1 s) + A_1(I)\cos(k_2\varphi + l_2 s),$$

where $A_0(I) = \dfrac{2\,\pi\,(k_1 I + l_1)\,a_0}{\sinh(\frac{(k_1 I + l_1)\,\pi}{2})}$ and $A_1 = \dfrac{2\,(k_2 I + l_2)\pi\,a_1}{\sinh(\frac{(k_2 I + l_2)\pi}{2})}$.

Definition

Reduced Poincaré function is

$$\mathcal{L}^*(I,\theta) = \mathcal{L}(I, \varphi - I\,\tau^*(I,\varphi,s), s - \tau^*(I,\varphi,s)),$$

where $\theta = \varphi - I\,s$.

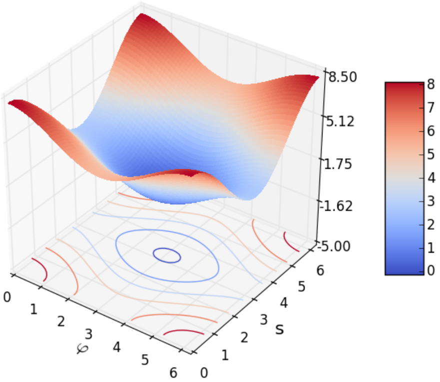


Figure: The Melnikov Potential, $\mu = a_0/a_1 = 0.6$, $I = 1$, $k_1 = l_2 = 1$ and $k_2 = l_1 = 0$.

Intersection point between invariant manifolds:

We look for τ^* such that $\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\,\tau^*, s - \tau^*) = 0.$

Different view-points of $\tau^*(I, \varphi, s)$

- Critical points of $\mathcal{L}$ on the straight line $R(I, \varphi, s) = \{(\varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$.
- Intersection between $R(I, \varphi, s) = \{(\varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$ and the crest which it is the curve of equation

$$\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\tau, s - \tau)|_{\tau=0} = 0.$$

Crests

Definition - Crests (Delshams-Huguet 2011)

For each I, we call *crests* $\mathcal{C}(I)$ the pair (φ, s) such that $\tau^* = 0$ satisfies

$$\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\,\tau^*, s - \tau^*) = 0. \qquad (3)$$

For the computation of the reduced Poincaré function, we have to study this equation.

- $(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π) always belong to the crest. One maximum and one minimum point and two saddle points.
- $\mathcal{L}^*(I,\theta)$ is $\mathcal{L}$ evaluated on the crest.
- $\theta = \varphi - Is$ is constant on the straight line $R(I,\varphi,s)$

Geometrical interpretation of the crest

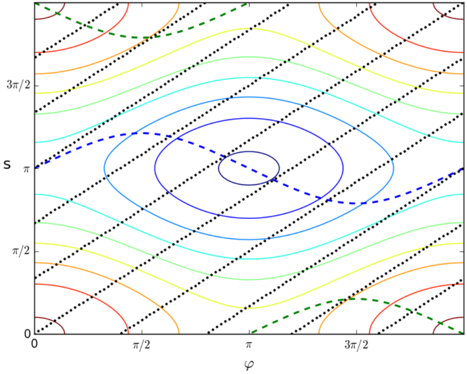


Figure: Level curves of $\mathcal{L}$ for $\mu = a_0/a_1 = 0.5$, $I = 1.2$, $k_1 = l_2 = 1$ and $k_2 = l_1 = 0$.

Understanding the behavior of the crests

$$\Downarrow$$

Understanding the behavior of the Reduced Poincaré function

$$\Downarrow$$

Understanding the Scattering map

We only need study two cases:

- The first (easier) case proven in *Regul. Chaotic Dyn.*

$$h(q, \varphi, s) = \cos q \left(a_0 \cos \varphi + a_1 \cos s \right)$$

- The second (more complicated) case, in progress

$$h(q, \varphi, s) = \cos q \left(a_0 \cos \varphi + a_1 \cos(\varphi - s) \right)$$

Each case has its own characteristics and together are enough to understand the general case.
We present just some highlights about each case.

Special Pseudo orbits: Highways for the first case

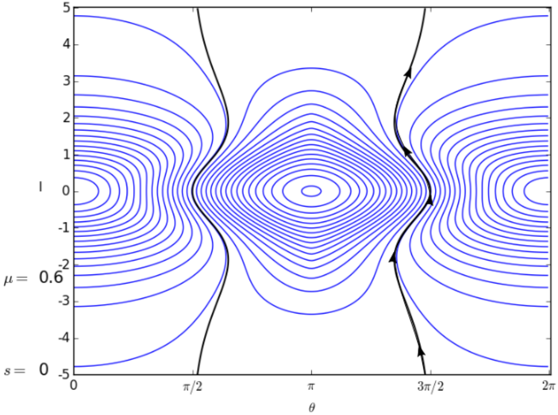
Definition: Highways

Highways are the level curves of $\mathcal{L}^*$ such that

$$\mathcal{L}^*(I,\theta) = \frac{2\pi a_0}{\sinh(\pi/2)}.$$

- Highways are "vertical"
- We always have a "pair" of highways. One goes up, the other goes down (this depends on signal of a_0/a_1.)
- It is easy to construct pseudo-orbits where highways are defined.

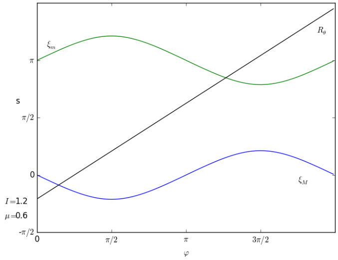
Special Pseudo orbits: Highways



$0 < |\mu| < 0.97$

- $|\mu\alpha(I)| < 1$, there are two crests $\mathcal{C}_{\mathrm{M,m}}(I)$ parameterized by:

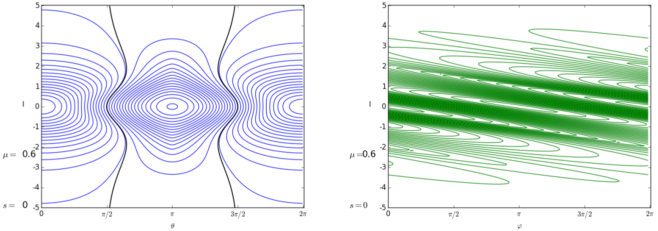
$$\begin{aligned} s = \xi_M(I,\varphi) &= -\arcsin(\alpha(I,\mu)\sin\varphi) \mod 2\pi \\ \xi_m(I,\varphi) &= \arcsin(\alpha(I,\mu)\sin\varphi) + \pi \mod 2\pi \end{aligned} \tag{4}$$



They are the horizontal crests

$0 < |\mu| < 0.625$

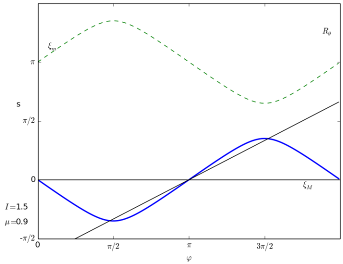
- For each I, the line $R(I, \varphi, s)$ and the crest $\mathcal{C}_{\mathrm{M,m}}(I)$ have only one intersection point.
- We have well defined S_M and S_m, where S_{M} is the scattering map associated to the intersections between $\mathcal{C}_{\mathrm{M}}(I)$ and $R(I, \varphi, s)$ and S_{m} is the scattering map associated to the intersection between $\mathcal{C}_{\mathrm{m}}(I)$ and $R(I, \varphi, s)$.

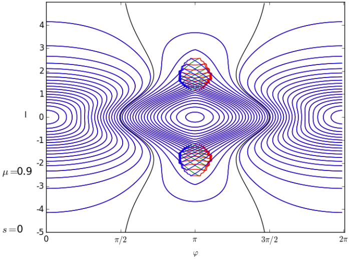


(a) Level curve of $\mathcal{L}^*_M(I, \theta)$. (b) Level curves of $\mathcal{L}^*_m(I, \theta)$

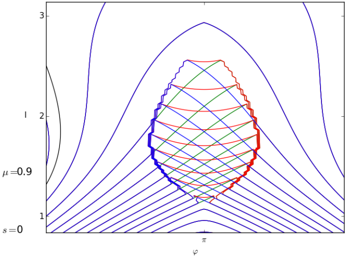
$0.625 < |\mu|$

- There are tangencies between $\mathcal{C}_{\mathrm{M,m}}(I, \varphi)$ and $R(I, \varphi, s)$. For some value of (I, φ, s), there are 3 points in $R(I, \varphi, s) \cap \mathcal{C}_{\mathrm{M,m}}(I)$.
- It implies that there are 3 scattering maps associated to each crest with different domains.(Multiple Scattering maps)





(c) The three types of level curves.



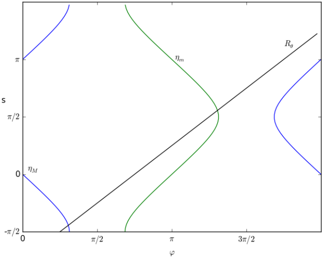
(d) Zoom where the scattering maps are different

Figure: Level curves of $\mathcal{L}^*_M(I,\theta)$, $\mathcal{L}^{*(1)}_M(I,\theta)$ and $\mathcal{L}^{*(2)}_M(I,\theta)$

$|\mu| > 0.97$

- For some values of I, $|\mu\alpha(I)| > 1$, the two crests $\mathcal{C}_{\mathrm{M,m}}$ are parameterized by:

$$\begin{aligned} \varphi = \eta_M(I,s) &= -\arcsin(\alpha(I,\mu)\sin s) \mod 2\pi \\ \eta_m(I,s) &= \arcsin(\alpha(I,\mu)\sin s) + \pi \mod 2\pi \end{aligned} \quad (5)$$



They are the vertical crests

As this happens for some values of I and when it happens, we can look this crests locally as the horizontal crests, we restrict the domain of the Scattering map.

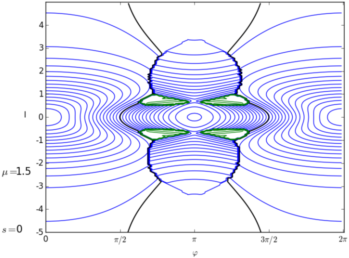


Figure: The level curves of $\mathcal{L}^*_{\mathsf{M}}(I, \theta)$, $\mu = 1.5$.

In green, the region where the scattering map S_{M} is not defined.

An example of pseudo-orbit

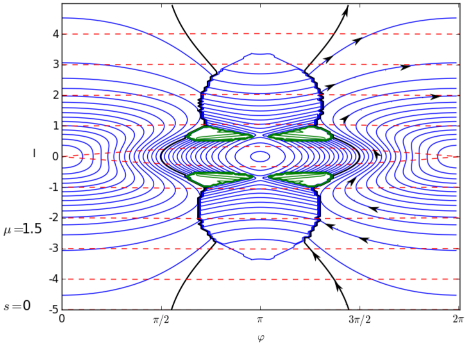


Figure: In red: Inner map, blue: Scattering map, black: Highways

Time of diffusion

An estimate of the total time of diffusion between I_0 and I_{f}, for simplicity only along the highways is

$$T_d \sim N_{\mathsf{s}} T_{\mathsf{h}} \sim \frac{T_{\mathsf{s}}}{\varepsilon} \log\left(\frac{C_{\mathsf{h}}}{\varepsilon}\right),$$

where

- $T_{\mathsf{h}} \approx \log\left(\frac{C_{\mathsf{h}}}{\varepsilon}\right)$ is the time along the homoclinic invariant manifold of $\widetilde{\Lambda}$, where $C_{\mathsf{h}} = 8\,|a_0|\left(1 + \dfrac{1.465}{\sqrt{1-\mu^2\alpha^2(I_{\mathsf{M}})}}\right)$
- $N_{\mathsf{s}} = T_{\mathsf{s}}/\varepsilon$ is the number of iterates of the scattering map along the highway and
- $T_s = \displaystyle\int_{I_0}^{I_f} \frac{-\sinh(I\pi/2)}{2\pi I a_0 \sin\psi_{\mathsf{h}}(I)} dI$, where $\psi_{\mathsf{h}} = \theta - I\tau^*(I,\theta)$ is a parametrization of the highway.

This estimate agrees with the optimal estimate of (Berti-Biasco-Bolle 2003) and (Treschev 2004), a time of the order $\mathcal{O}(\varepsilon^{-1}\log\varepsilon^{-1})$.

Main differences between the first and second cases

In the second case:

- There are no *Highways*.
- For any value of $\mu = a_0/a_1$ is possible to find I_{h} and I_{v} such that for I_{h} the crests are horizontal and for I_{v} the crests are vertical.
- For any value of μ there exists I such that the crests and $R(I, \varphi, s)$ are tangent.

Same crest, different scattering map

How to take $\tau^*(I, \theta)$ is very important and useful.
Green zones: I increases under scattering map.
Red zones: I decreases under scattering map.

Figure: "Lower" crest.

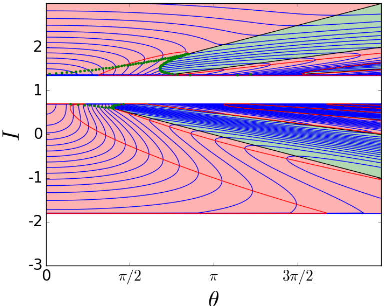


Figure: "Upper" crest

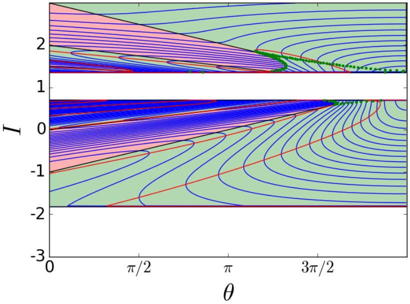
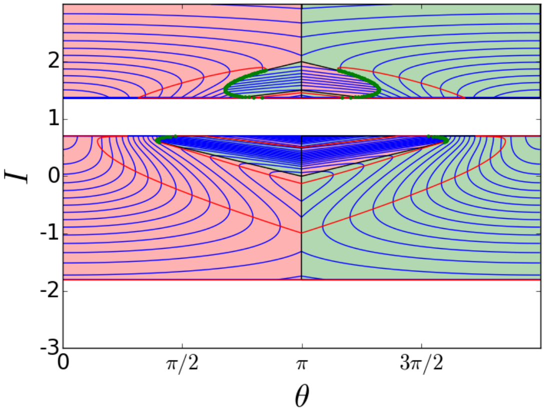
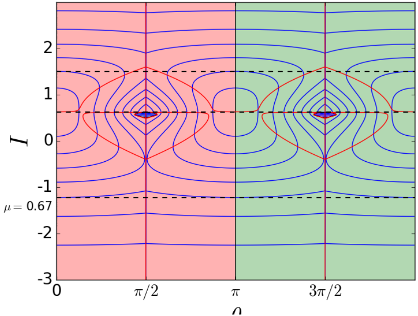


Figure: Lower $|\tau^*|$



Combination of Scattering maps: A non-smooth vector field

In this picture we show a combination of 6 scattering maps.



A Hamiltonian with $3 + 1/2$ dof

$$H(I_1, I_2, \varphi_1, \varphi_2, p, q, t, \varepsilon) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + h(I_1, I_2) + \varepsilon \cos q \, g(\varphi_1, \varphi_2, t),$$

where

$$h(I_1, I_2) = \Omega_1 \frac{I_1^2}{2} + \Omega_2 \frac{I_2^2}{2}$$

and

$$g(\varphi_1, \varphi_2, t) = a_1 \cos\varphi_1 + a_2 \cos\varphi_2 + a_3 \cos(\varphi_1 + \varphi_2 - t).$$

A Hamiltonian with $3 + 1/2$ dof

In this case, the Melnikov potential is

$$\mathcal{L}(I, \varphi - \omega\tau) = \sum_{i=1}^{3} A_i \cos(\varphi_i - \omega_i\tau),$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, $\omega = (\omega_1, \omega_2, \omega_3)$, $\varphi_3 = \varphi_1 + \varphi_2 - s$,

$$A_i = \frac{2\,\pi\omega_i}{\sinh(\frac{\pi\,\omega_i}{2})} a_i,$$

and

$$\omega_1 = \Omega_1 I_1 \quad \omega_2 = \Omega_2 I_2 \quad \omega_3 = \omega_1 + \omega_2 - 1.$$

Example of crests

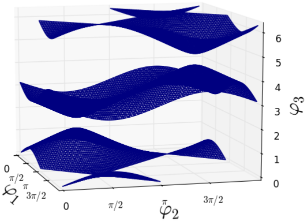


Figure: Horizontal crests: $\mu_1 = \mu_2 = 0.48$ $,\omega_1 = \omega_2 = 1.219.$

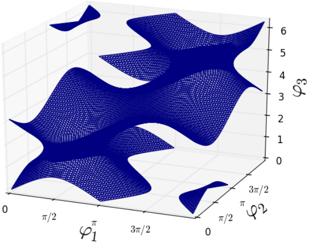


Figure: Crests with holes : $\mu_1 = 0.7, \mu_2 = 0.6$ $,\omega_1 = \omega_2 = 1.219.$

Behavior of the crests

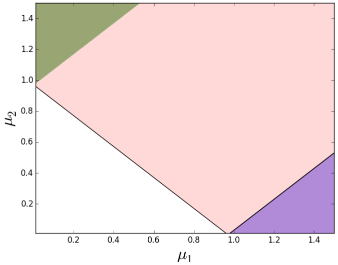


Figure: $\omega_1 = \omega_2 = 1.219$

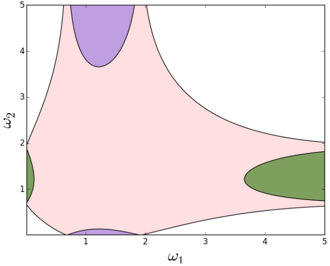


Figure: $\mu_1 = \mu_2 = 1.2$

Pink: Surface with holes, white: horizontal surfaces $s(\varphi_1, \varphi_2)$, purple: vertical surfaces $\varphi_1(\varphi_2, s)$, green: vertical surfaces $\varphi_2(\varphi_1, s)$.

Muchas gracias!

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