Scattering maps and Global Instability in Hamiltonian Systems

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Global instability

What is Global instability in Hamiltonian systems?

Assume a Hamiltonian system given by the Hamiltonian:

$$H(q,p,I,\varphi) = h_0(q,p,I) + \varepsilon h_1(q,p,I,\varphi,t). \tag{1}$$

For $\varepsilon = 0$,

$$\dot{I} = \frac{\partial h_0}{\partial \varphi} = 0 \Rightarrow I = \text{constant}. \tag{2}$$

There exists a global instability in the variable I if for a $\varepsilon \neq 0$, there exists an orbit of the system (1) such that

$$\triangle I := I(T) - I(0) = \mathcal{O}(1). \tag{3}$$

This instability is also called Arnold diffusion.

Consider a pendulum and a rotor plus a time periodic perturbation depending on two harmonics in the variables (φ, s):

$$H_\varepsilon(p,q,I,\varphi,s) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2} + \varepsilon h(q,\varphi,s) \tag{4}$$

$$\begin{gathered} h(q,\varphi,s) = f(q)g(\varphi,s), \\ f(q) = \cos q, \qquad g(\varphi,s) = a_1\cos(k_1\varphi + l_1 s) + a_2\cos(k_2\varphi + l_2 s), \end{gathered} \tag{5}$$

with $k_1,\ k_2,\ l_1,\ l_2 \in \mathbb{Z}$.

Theorem

Assume that $a_1a_2 \neq 0$ *and* $\begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix} \neq 0$ *in* (4)-(5). *Then, for any* $I^* > 0$, *there exists* $\varepsilon^* = \varepsilon^*(I^*, a_1, a_2) > 0$ *such that for any* ε, $0 < \varepsilon < \varepsilon^*$, *there exists a trajectory* $(p(t), q(t), I(t), \varphi(t))$ *such that for some* $T > 0$

$$I(0) \leq -I^* < I^* \leq I(T).$$

It is easy to check that if

$$\Delta := k_1 l_2 - k_2 l_1 = 0 \quad \text{or} \quad a_1 = 0 \quad \text{or} \quad a_2 = 0$$

there is no global instability for the variable I.

If $\Delta a_1 a_2 \neq 0$, after some rational linear changes in the angles, we only need to study two cases:

- The first (and easier) case [Delshams-S17]

$$g(\varphi, s) = a_1 \cos \varphi + a_2 \cos s$$

- The second case [Delshams-S17a]

$$g(\varphi, \sigma) = a_1 \cos \varphi + a_2 \cos \sigma,$$

where $\sigma = \varphi - s$.

We deal with an a priori unstable Hamiltonian [Chierchia-Gallavotti94].

In the unperturbed case $\varepsilon = 0$, the Hamiltonian H_0 is integrable formed by the standard pendulum plus a rotor

$$H_0(p, q, I, \varphi, s) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2}.$$

I is constant: $\triangle I := I(T) - I(0) \equiv 0.$

For any $0 < \varepsilon \ll 1$, there is a finite drift in the action of the rotor I: $\triangle I = \mathcal{O}(1)$, so we have global instability.

Basically, we ensure the Arnold diffusion performing the following scheme:

- To construct iterates under several Scattering maps and the Inner map, giving rise to diffusing pseudo-orbits.
- To use previous results about Shadowing [Fontich-Martín00], [Gidea-Llave-Seara14] for ensuring the existence of real orbits close to the pseudo-orbits.

We have two important dynamics associated to the system: the inner and the outer dynamics on a large invariant object $\widetilde{\Lambda}$.

$$\widetilde{\Lambda} = \{(0, 0, I, \varphi, s); I \in [-I^*, I^*], (\varphi, s) \in \mathbb{T}^2\}.$$

is a 3D *Normally Hyperbolic Invariant Manifold* (NHIM) with associated 4D stable $W^s_\varepsilon(\widetilde{\Lambda})$ and unstable $W^u_\varepsilon(\widetilde{\Lambda})$ invariant manifolds.

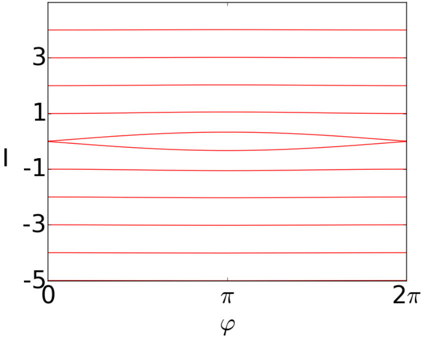
- The *inner dynamics* is the dynamics restricted to $\widetilde{\Lambda}$. (Inner map)
- The *outer dynamics* is the dynamics along the invariant manifolds of $\widetilde{\Lambda}$. (Scattering map)

Remark: Due to the form of the perturbation, $\widetilde{\Lambda} = \widetilde{\Lambda}_\varepsilon$ (not essential).

For the first case $g(\varphi, s) = a_1 \cos\varphi + a_2 \cos s$, the inner dynamics is described by the Hamiltonian system with the Hamiltonian

$$K(I, \varphi, s) = \frac{I^2}{2} + \varepsilon \left(a_1 \cos\varphi + \cancel{a_2 \cos s} \right).$$

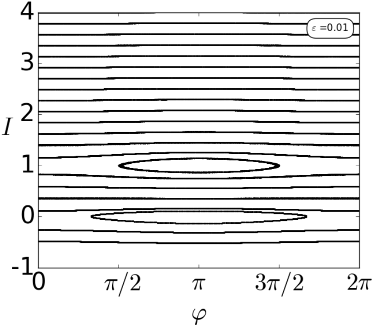
In this case the inner dynamics is integrable.



For $g(\varphi, \sigma)$, the inner dynamics is described by the Hamiltonian

$$K(I, \varphi, \sigma) = \frac{I^2}{2} + \varepsilon \left(a_1 \cos \varphi + a_2 \cos \sigma \right),$$

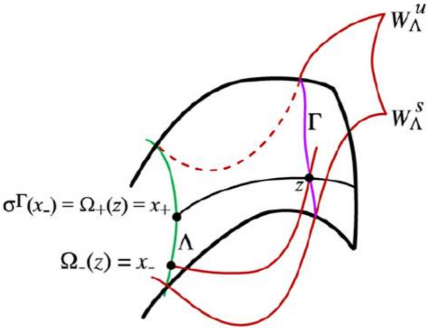
where $\sigma = \varphi - s$. The system associated to this Hamiltonian is not integrable and two resonances arise in $I = 0$ and $I = 1$.



Let $\widetilde{\Lambda}$ be a NHIM with invariant manifolds intersecting transversally along a homoclinic manifold Γ. A scattering map is a map S defined by $S(\tilde{x}_-) = \tilde{x}_+$ if there exists $\tilde{z} \in \Gamma$ satisfying

$$|\phi_t^\varepsilon(\tilde{z}) - \phi_t^\varepsilon(\tilde{x}_\mp)| \longrightarrow 0 \text{ as } t \longrightarrow \mp\infty$$

that is, $W_\varepsilon^u(\tilde{x}_-)$ intersects transversally $W_\varepsilon^s(\tilde{x}_+)$ in $\tilde{z}$.



S is an exact symplectic map [Delshams-Llave-Seara08] and takes the form:

$$\mathcal{S}_\varepsilon(I,\theta) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I,\theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I,\theta) + \mathcal{O}(\varepsilon^2) \right),$$

where $\theta = \varphi - Is$ and $\mathcal{L}^*(I,\theta)$ is the Reduced Poincaré function.

- The variable s remains fixed under S_ε: it plays the role of a parameter
- Up to first order in ε, $\mathcal{S}_\varepsilon$ is the $-\varepsilon$-time flow of the Hamiltonian $\mathcal{L}^*(I,\theta)$
- The scattering map jumps $\mathcal{O}(\varepsilon)$ distances along the level curves of $\mathcal{L}^*(I,\theta)$

Now, we are going to construct the Reduced Poincaré function $\mathcal{L}^*$.

To get a scattering map we search for homoclinic orbits to $\tilde{\Lambda}_\varepsilon$

Proposition

Given $(I, \varphi, s) \in [-I^*, I^*] \times \mathbb{T}^2$, assume that the real function

$$\tau \in \mathbb{R} \longmapsto \mathcal{L}(I, \varphi - I\,\tau, s - \tau) \in \mathbb{R}$$

has a non degenerate critical point $\tau^* = \tau(I, \varphi, s)$, where

$$\mathcal{L}(I, \varphi, s) = \int_{-\infty}^{+\infty} (\cos q_0(\sigma) - \cos 0)\, g(\varphi + I\sigma, s + \sigma; 0) d\sigma.$$

Then, for $0 < |\varepsilon|$ small enough, there exists a transversal homoclinic point $\tilde{z}$ to $\widetilde{\Lambda}_\varepsilon$, which is ε-close to the point $\tilde{z}^*(I, \varphi, s) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) \in W^0(\widetilde{\Lambda})$:

$$\tilde{z} = \tilde{z}(I, \varphi, s) = (p_0(\tau^*) + O(\varepsilon), q_0(\tau^*) + O(\varepsilon), I, \varphi, s) \in W^u(\widetilde{\Lambda}_\varepsilon) \pitchfork W^s(\widetilde{\Lambda}_\varepsilon).$$

In our model $q_0(t) = 4 \arctan \mathrm{e}^t$, $p_0(t) = 2/\cosh t$ is the separatrix for positive p of the standard pendulum $P(q,p) = p^2/2 + \cos q - 1$.

- For $g(\varphi, s) = a_1 \cos\varphi + a_2 \cos s$, the Melnikov potential becomes

$$\mathcal{L}(I,\varphi,s) = A_1(I)\cos\varphi + A_2 \cos s,$$

where $A_1(I) = \dfrac{2\,\pi\, I\, a_1}{\sinh\left(\frac{I\pi}{2}\right)}$ and $A_2 = \dfrac{2\,\pi\, a_2}{\sinh\left(\frac{\pi}{2}\right)}$.

- For $g(\varphi, \sigma) = a_1 \cos\varphi + a_2 \cos\sigma$ $(\sigma = \varphi - s)$, the Melnikov potential becomes

$$\mathcal{L}(I,\varphi,\sigma) = A_1(I)\cos\varphi + A_2(I)\cos\sigma,$$

where $A_1(I)$ is as before but now $A_2(I) = \dfrac{2\,(I-1)\pi\, a_2}{\sinh\left(\frac{(I-1)\pi}{2}\right)}$.

Finally, the function $\mathcal{L}^*(I,\theta)$ can be defined:

Definition

The Reduced Poincaré function is

$$\mathcal{L}^*(I,\theta) = \mathcal{L}(I, \varphi - I\,\tau^*(I,\varphi,s), s - \tau^*(I,\varphi,s)),$$

where $\theta = \varphi - I\,s$.

Therefore the definition of $\mathcal{L}^*(I, \theta = \varphi - Is)$ depends on the function $\tau^*(I,\varphi,s)$.
So, we need to calculate τ^* to obtain the $\mathcal{L}^*$.

From the Proposition given above, we look for τ^* such that $\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\,\tau^*, s - \tau^*) = 0$.

Different view-points for $\tau^* = \tau^*(I, \varphi, s)$

- Look for critical points of $\mathcal{L}$ on the straight line, called NHIM line $R(I, \varphi, s) = \{(I, \varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$.
- Look for intersections between $R(I, \varphi, s) = \{(I, \varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$ and a crest which is a curve of equation
$$\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\tau, s - \tau)|_{\tau=0} = 0.$$

Note that the crests are characterized by $\tau^*(I, \varphi, s) = 0$.
The crests were introduced in [Delshams-Huguet11]. A similar construction appears in [Davletshin-Treschev16].

Definition - Crests [Delshams-Huguet11]

For each I, we call *crest* $\mathcal{C}(I)$ the set of curves in the variables (φ, s) of equation

$$I\frac{\partial \mathcal{L}}{\partial \varphi}(I,\varphi,s) + \frac{\partial \mathcal{L}}{\partial s}(I,\varphi,s) = 0. \tag{6}$$

which in our case can be rewritten as

$$g(\varphi,s)\text{:}\ \ \mu\alpha(I)\sin\varphi + \sin s = 0, \qquad \text{with } \alpha(I) = \frac{I^2\sinh(\frac{\pi}{2})}{\sinh(\frac{\pi I}{2})}, \quad \mu = \frac{a_1}{a_2}.$$

$$g(\varphi,\sigma=\varphi-s)\text{:}\ \ \mu\alpha(I)\sin\varphi + \sin\sigma = 0, \qquad \text{with } \alpha(I) = \frac{I^2\sinh(\frac{(I-1)\pi}{2})}{(I-1)^2\sinh(\frac{\pi I}{2})}, \quad \mu = \frac{a_1}{a_2}.$$

- For any I, the critical points of the Melnikov potential $\mathcal{L}(I,\cdot,\cdot)$ ($(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π): one maximum, one minimum point and two saddle points) always belong to the crest $\mathcal{C}(I)$.
- $\mathcal{L}^*(I,\theta)$ is nothing else but $\mathcal{L}$ evaluated on the crest $\mathcal{C}(I)$.
- $\theta = \varphi - Is$ is constant on the NHIM line $R(I,\varphi,s)$

Understanding the behavior of the crests

$$\Downarrow$$

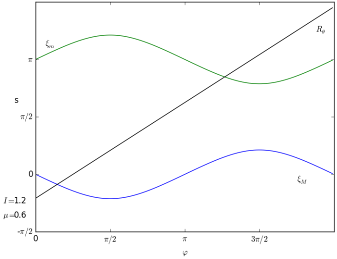
Understanding the behavior of the Reduced Poincaré function

$$\Downarrow$$

Understanding the Scattering map

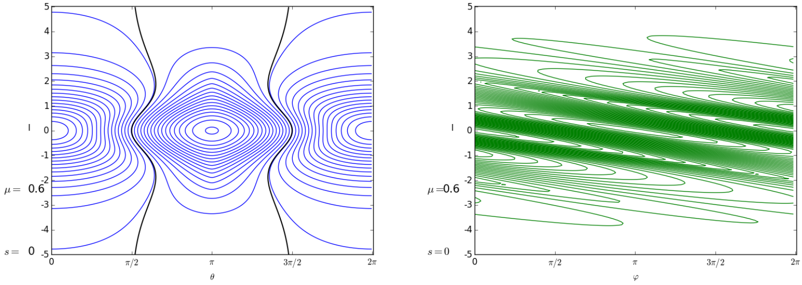
- For $|\mu\alpha(I)| < 1$, there are two crests $\mathcal{C}_{\mathrm{M,m}}(I)$ parameterized by:

$$\begin{aligned} s = \xi_M(I,\varphi) &= -\arcsin(\mu\alpha(I)\sin\varphi) \quad \mod 2\pi \\ \xi_m(I,\varphi) &= \arcsin(\mu\alpha(I)\sin\varphi) + \pi \quad \mod 2\pi \end{aligned} \tag{7}$$



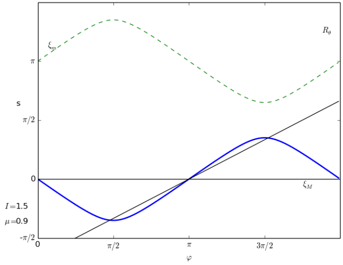
They are "horizontal" crests

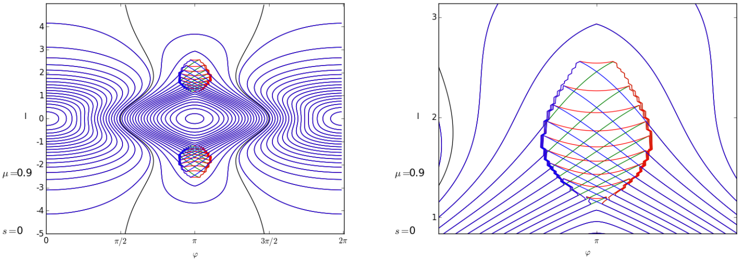
- For each I, the NHIM line $R(I, \varphi, s)$ and the crest $\mathcal{C}_{\mathrm{M,m}}(I)$ has only one intersection point.
- The scattering map S_{M} associated to the intersections between $\mathcal{C}_{\mathrm{M}}(I)$ and $R(I, \varphi, s)$ is well defined for any $\varphi \in \mathbb{T}$. Analogously for S_{m}, changing M to m. In the variables $(I, \theta = \varphi - Is)$, both scattering maps $\mathcal{S}_{\mathrm{M}}$, $\mathcal{S}_{\mathrm{m}}$ are globally well defined.



(a) Level curves of $\mathcal{L}^*_M(I, \theta)$ (b) Level curves of $\mathcal{L}^*_m(I, \theta)$

- There are tangencies between $\mathcal{C}_{\mathrm{M,m}}(I, \varphi)$ and $R(I, \varphi, s)$. For some value of (I, φ, s), there are 3 points in $R(I, \varphi, s) \cap \mathcal{C}_{\mathrm{M,m}}(I)$.
- This implies that there are 3 scattering maps associated to each crest with different domains.(Multiple Scattering maps)





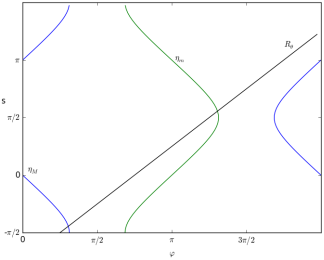
(c) The three types of level curves.

(d) Zoom where the scattering maps are different

Figure: Level curves of $\mathcal{L}^*_M(I,\theta)$, $\mathcal{L}^{*(1)}_M(I,\theta)$ and $\mathcal{L}^{*(2)}_M(I,\theta)$

- For some values of I, $|\mu\alpha(I)| > 1$, the two crests $\mathcal{C}_{\mathrm{M,m}}$ are parameterized by:

$$\begin{aligned} \varphi = \eta_M(I,s) &= -\arcsin(\mu\alpha(I)\sin s) \qquad \mod 2\pi \\ \eta_m(I,s) &= \arcsin(\mu\alpha(I)\sin s) + \pi \qquad \mod 2\pi \end{aligned} \tag{8}$$



They are "vertical" crests

For the values of I for which horizontal crests become vertical, it is not always possible to prolong in a continuous way the scattering maps, so the domain of the scattering map has to be restricted.

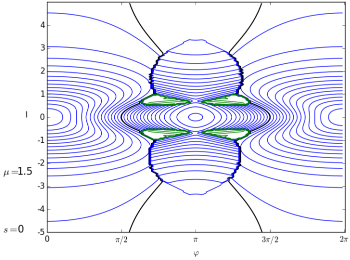


Figure: The level curves of $\mathcal{L}^*_{\mathrm{M}}(I, \theta)$, $\mu = 1.5$.

In green, the region where the scattering map S_{M} is not defined.

Definition: Highways

Highways are the level curves of $\mathcal{L}^*$ such that

$$\mathcal{L}^*(I, \theta) = \frac{2\pi a_1}{\sinh(\pi/2)}.$$

- The highways are "vertical" in the variables (φ, s)
- We always have a pair of highways. One goes up, the other goes down (this depends on the sign of $\mu = a_1/a_2$)
- The highways give rise to fast diffusing pseudo-orbits

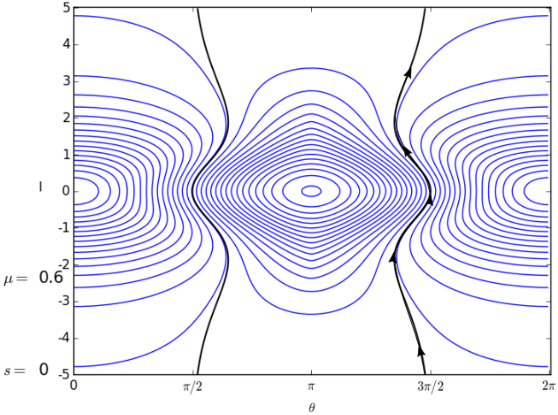


Figure: The scattering map jumps $\mathcal{O}(\varepsilon)$ distances along the level curves of $\mathcal{L}^*(I, \theta)$

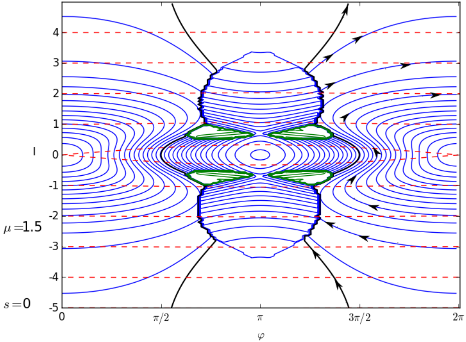


Figure: In red: Inner map, blue: Scattering map, black: Highways, $\mu = 1.5$.

An estimate of the total time of diffusion between $-I^*$ and I^*, close to the highway, is

$$T_\mathrm{d} = \frac{T_\mathrm{s}}{\varepsilon}\left[2\log\left(\frac{C}{\varepsilon}\right) + \mathcal{O}(\varepsilon^b)\right], \text{ for } \varepsilon \to 0, \text{ where } 0 < b < 1,$$

with

$$T_\mathrm{s} = T_\mathrm{s}(I^*, a_1, a_2) = \int_0^{I^*} \frac{-\sinh(\pi I/2)}{\pi a_1 I \sin\psi_\mathrm{h}(I)}\, dI,$$

where $\psi_\mathrm{h} = \theta - I\tau^*(I,\theta)$ is the parameterization of the highway $\mathcal{L}^*(I,\psi_\mathrm{h}) = A_2$, and

$$C = C(I^*, a_1, a_2) = 16\,|a_1|\left(1 + \frac{1.465}{\sqrt{1-\mu^2 A^2}}\right)$$

where $A = \max_{I\in[0,I^*]} \alpha(I)$, with $\alpha(I) = \frac{\sinh(\frac{\pi}{2})\, I^2}{\sinh(\frac{\pi I}{2})}$ and $\mu = a_1/a_2$.

Note: This estimate agrees with the upper bounds given in [Bessi-Chierchia-Valdinoci01] and quantifies the general optimal diffusion estimate $\mathcal{O}\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ of [Berti-Biasco-Bolle03] and [Treschev04].

Second case: $g(\varphi, \sigma)$, $\sigma = \varphi - s$

Now we describe the case which the perturbation takes the form

$$h(\varphi, \sigma) = \cos q \left(a_1 \cos \varphi + a_2 \cos \sigma \right),$$

where $\sigma = \varphi - s$.

In the second case:

- For $|\mu\alpha(I)| < 1$, there are two crests $\mathcal{C}_{\mathrm{M,m}}(I)$ parameterized by $\sigma = \xi_M(I, \varphi)$ and $\xi_m(I, \varphi)$. For $|\mu\alpha(I)| > 1$, $\mathcal{C}_{\mathrm{M,m}}(I)$ parameterized by $\varphi = \eta_M(I, \sigma)$ and $\eta_m(I, \sigma)$. The crests lie on the plane (φ, σ)
- There are no *Highways*.
- For any value of $\mu = a_1/a_2$ is possible to find I_{h} and I_{v} such that for $I = I_{\mathrm{h}}$ the crests are horizontal and for $I = I_{\mathrm{v}}$ the crests are vertical.
- For any value of μ there exists I such that the crests and some NHIM line are tangent. There are always multiple scattering maps

The choice of the concrete curve of the crest and therefore of $\tau^*(I, \theta)$ is very important and useful.

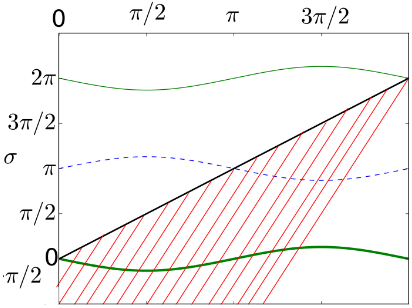


Figure: Going down along NHIM lines

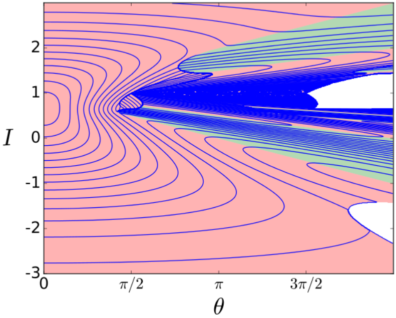


Figure: The "lower" crest

Green zones: I increases under the scattering map.

Red zones: I decreases under the scattering map.

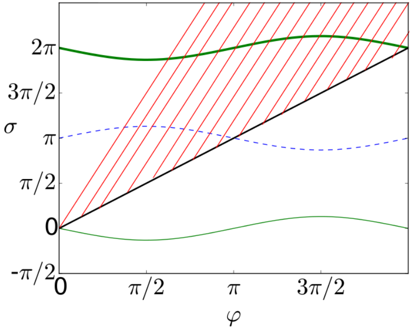


Figure: Going up along NHIM lines

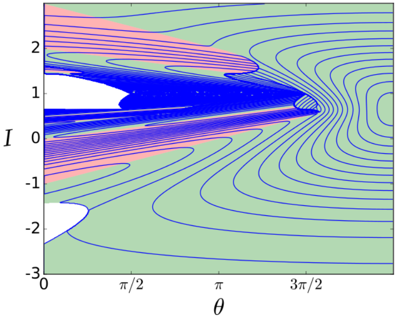


Figure: The "upper" crest

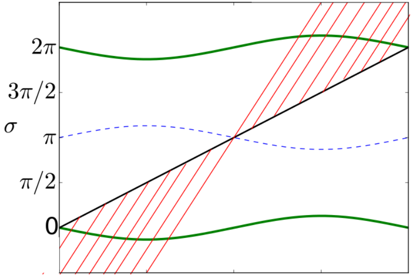


Figure: Minimal time

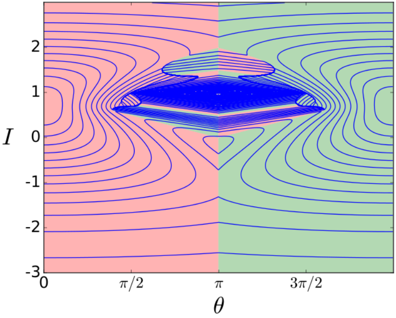


Figure: Minimal $|\tau^*|$ between "lower" and "upper" crest

In this picture we show a combination of 3 scattering maps.

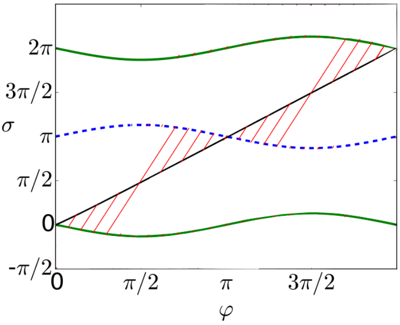


Figure: First intersection

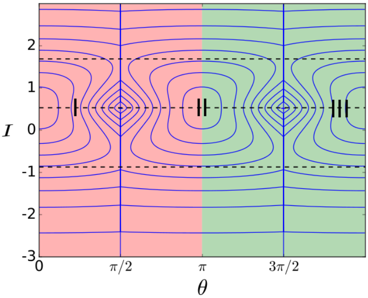


Figure: Minimal $|\tau^*|$ between $\mathcal{C}_{\mathrm{M}}(I)$ and $\mathcal{C}_{\mathrm{m}}(I)$

Thank you very much.

Muchas gracias.

Moltes gràcies.

Muito obrigado.

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