Lecture 4: Arnold Diffusion in Celestial Mechanics

Arnold Diffusion and applications

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We will study the paper

Global Instability in the Restricted Planar Elliptic Three Body Problem

by Delshams, Kaloshin, de la Rosa and Seara.

Consider the motion of a massless (a comet) particle q under the attraction of two massive bodies $q_{\mathbf{S}}$ and $q_{\mathbf{J}}$ with masses $m_{\mathbf{S}} = 1 - \mu$ and $m_{\mathbf{J}} = \mu$, respectively, which move in elliptic orbits with eccentricity $e_{\mathbf{J}}$ around their center of mass.

$q_{\mathbf{S}}$ and $q_{\mathbf{J}}$ are called primaries (Sun and Jupiter,respectively).

Denote by $G = q \times \dot{q}$ the angular momentum of the particle q, in the paper, the authors proved that there exist solution with a large variation (diffusion) of G.

Precisely, the following theorem:

Theorem

There exist two constants $C > 0$, $c > 0$ *such that for any* $0 < e_J < c/C$ *there is* $\mu^* = \mu^*(C, c, e_J) > 0$ *such that for any* $0 < \mu < \mu^*$ *and any* $C \leq G_1^* < G_2^* \leq c/e_J$ *there exists a trajectory of the RPETBP such that* $G(0) < G_1^*$, $G(T) > G_2^*$ *for some* $T > 0$.

From the gravitational Newton's law

$$\frac{d^2q}{dt^2} = (1-\mu)\frac{q_S - q}{|q_S - q|^3} + \mu\frac{q_J - q}{|q_J - q|^3}$$

By introduce $p = dq/dt$, we can rewrite as a $2 + 1/2$ degrees of freedom Hamiltonian systems

$$H_\mu(q, p, t; e_J) = \frac{p^2}{2} - U_\mu(q, t; e_J),$$

where

$$U_\mu(q, t; e_J) = \frac{1-\mu}{|q - q_S|} + \frac{\mu}{|q - q_J|}.$$

By writing the system in polar coordinates:

$$q = \rho(\cos\alpha, \sin\alpha), \quad q_S = \mu r(\cos f, \sin f), \quad q_J = -(1-\mu) r(\cos f, \sin f),$$

where r is the distance between the primary bodies and $f(t, e_J)$ is called the true anomaly, and more

$$r = \frac{1-e_J^2}{1+e_J\cos f} \quad \text{and} \quad \frac{df}{dt} = \frac{(1+e_J\cos f)^2}{(1-e_J)^{3/2}}$$

In the new coordinates, the Hamiltonian takes the form

$$H_\mu^*(\rho, \alpha, y, G, t; e_J) = \frac{y^2}{2} + \frac{G^2}{2\rho^2} - U_\mu^*(\rho, \alpha, t; e_J),$$

where

$$U_\mu^* = \frac{1-\mu}{\sqrt{\rho^2 - 2\mu r \rho \cos(\alpha - f) + \mu^2 r^2}} + \frac{\mu}{\sqrt{\rho^2 + 2(1-\mu) r \rho \cos(\alpha - f) + (1-\mu)^2 r^2}}$$

For $e_J = 0$, $r = 1$ and $\frac{df}{dt} = 1$, then $f = t$. This is the circular case and by taking a new angle $\alpha - t$ we have that this is a 2 d.o.f Hamiltonian. (there is no diffusion).

To study the behavior of the solutions close to $\rho = \infty$, we use the non canonical McGehee coordinates

$$\rho = \frac{2}{x^2}.$$

The previous Hamiltonian becomes into

$$\mathcal{H}_\mu(x, \alpha, y, G, t; e_J) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \underbrace{\frac{x^2}{2}\left(\frac{1-\mu}{\sigma_S} + \frac{\mu}{\sigma_J}\right)}_{\mathcal{U}_\mu(x,\alpha,t,e_J)},$$

where

- $\sigma_S^2 = 1 - \mu r x^2 \cos(\alpha - f) + \frac{\mu^2 r^2 x^4}{4}$
- $\sigma_J^2 = 1 + (1-\mu) r x^2 \cos(\alpha - f) + \frac{(1-\mu)^2 r^2 x^4}{4}$

The differential equations in these coordinates we have

$$\frac{dx}{dt} = -\frac{x^3}{4}\frac{\partial \mathcal{H}_\mu}{\partial y} \qquad \frac{dy}{dt} = \frac{-x^3}{4}\left(-\frac{\partial \mathcal{H}_\mu}{\partial x}\right)$$

$$\frac{d\alpha}{dt} = \frac{\partial \mathcal{H}_\mu}{\partial G} \qquad \frac{dG}{dt} = -\frac{\partial \mathcal{H}_\mu}{\partial \alpha}.$$

For $\mu = 0$, the above system represents the Kepler problem.

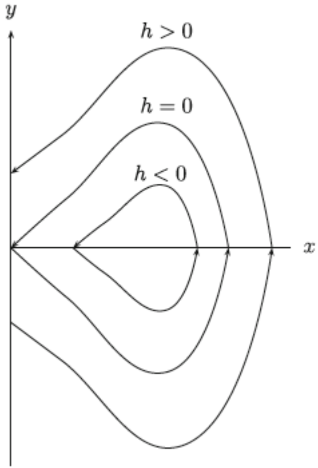
$$\mathcal{H}_0(x, y, G) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \frac{x^2}{2}$$

whith differential equations

$$\frac{dx}{dt} = \frac{-x^3 y}{4} \qquad \frac{dy}{dt} = \frac{G^2 x^6}{8} - \frac{x^4}{4}$$

$$\frac{d\alpha}{dt} = \frac{x^4 G}{4} \qquad \frac{dG}{dt} = 0.$$

- G is conserved.
- At $(x, y) = (0, 0)$, α and G are constants
- $\Lambda_{\alpha,G} = \{(0, \alpha, 0, G)\}$ is a parabolic equilibrium point (has the linear part equal to zero) and has 1D homoclinic invariant manifold $\gamma_{\alpha,G} = W^{u}(\Lambda_{\alpha,G}) = W^{s}(\Lambda_{\alpha,G})$


y
h > 0
h = 0
h < 0
x

Then,

$$\Lambda_\infty = \bigcup_{\alpha, G} \Lambda_{\alpha, G}$$

is a 2D manifold of parabolic equilibrium points.

By extending the phase space (we can consider the time $s \in \mathbb{T}$ and $ds/dt = 1$):

$$\tilde{\Lambda}_\infty = \{(0, \alpha, 0, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\} \quad (3D)$$

with invariant stable and unstable manifold given by

$$\tilde{\gamma} = \bigcup_{\alpha, G} \tilde{\gamma}_{\alpha, G} = W^{u}(\tilde{\Lambda}_\infty) = W^{s}(\tilde{\Lambda}_\infty)$$
$$= \{(x, \alpha, y, G, s) : (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}, \mathcal{H}_0 = 0\} \quad (4D).$$

$\Rightarrow \tilde{\Lambda}_\infty$ is *Topologically equivalent to a Normally Invariant Manifold.* TNHIM.

For $\mu > 0$, we can write $\mathcal{H}_\mu$ as

$$\mathcal{H}_\mu(x, \alpha, y, G, s; e_J) = \mathcal{H}_0(x, y, G) - \mu\Delta\mathcal{U}_\mu(x, \alpha, s; e_J),$$

where $\Delta\mathcal{U}_\mu(x, \alpha, s; e_J) := \mathcal{U}_\mu(x, \alpha, s; e_J) - x^2/2$.

Therefore, it is possible to study $\mathcal{H}_\mu$ as a perturbation of the Kepler problem ($\mu = 0$).

- $\tilde{\Lambda}_\infty$ remains invariant for $\mu > 0$ and all the periodic orbits $\tilde{\Lambda}_{\alpha,G}$ persist.
- The inner dynamics (the dynamics restricted to $\tilde{\Lambda}_\infty$) is trivial, since it consists of fixed points.Only the time varies.
- For $\mu \ll 1$, $W^u(\tilde{\Lambda}_\infty)$ and $W^s(\tilde{\Lambda}_\infty)$ exist, but they do not longer coincide.

The strategy in this case is very similar to the strategy applied in the last lecture's example, replacing the NHIM by TNHIM.

- To use the Melnikov Potential to find the transverse intersections between $W^u(\tilde{\Lambda}_\infty)$ and $W^s(\tilde{\Lambda}_\infty)$.
- To define at least two scattering maps by using the found intersections in the previous step.
- To combine them in order to obtain a pseudo-orbit that presents a diffusion on action variable G.
- To apply a suitable shadowing lemma to ensure the existence of a real orbit "close" to the pseudo-orbit.

Remark: As the inner dynamics are fixed points, they are not useful for diffusion paths. This is the reason that we have to combine different scattering maps.

The Melnikov potential is

$$\mathcal{L}(\alpha, G, s; e_J) = \int_{-\infty}^{\infty} \Delta\mathcal{U}_0(x_h(t; G), \alpha_h(t; \alpha, G), s + t; e_J)dt,$$

where x_h, α_h are a parametrization of the homoclinic manifold ($\mu = 0$) and $\Delta\mathcal{U}_0 = \lim_{\mu\to 0} \Delta\mathcal{U}_\mu$

Proposition

Given $(\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}$, assume that the function

$$\tau \in \mathbb{R} \to \mathcal{L}(\alpha, G, s - \tau; e_J)$$

has a non-degenerate critical point $\tau^* = \tau^*(\alpha, G, s; e_J)$. Then, there exists $\mu^* = \mu^*(G, e_J)$, such that for $0 < \mu < \mu^*$,close to the point $\tilde{\mathbf{z}}_0^* \in \tilde{\gamma}$, there exists a locally unique point $\tilde{\mathbf{z}}^* \in W^{\mathrm{u}}(\tilde{\Lambda}_\infty) \pitchfork W^{\mathrm{s}}(\tilde{\Lambda}_\infty) \pitchfork N(\tilde{\mathbf{z}}_0^*)$ of the form

$$\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}_0^* + \mathcal{O}(\mu).$$

Also, there exist unique points
$\tilde{\mathbf{x}}_\pm = (0, \alpha_\pm, 0) = (0, \alpha, 0, G, s) + \mathcal{O}(\mu) \in \tilde{\Lambda}_\infty$ such that

$$\tilde{\phi}_{t,\mu}(\tilde{z}^*) - \tilde{\phi}_{t,\mu}(\tilde{\mathbf{x}}_\pm) \to 0 \quad as \quad t \to \pm\infty.$$

The last property says that we can define a scattering map

$$\tilde{\mathbf{x}}_+ = S(\tilde{\mathbf{x}}_-)$$

Once we have $\tau^*(\alpha, G, s; e_J)$, we can define the Reduced Poincaré function

$$\mathcal{L}^*(\alpha, G, e_J) := \mathcal{L}(\alpha, G, s - \tau^*).$$

And the Scattering map formula is

$$\mathcal{S}_\mu(\alpha, G, s) = (\alpha - \mu \frac{\partial \mathcal{L}^*}{\partial G}(\alpha, G, e_J) + \mathcal{O}(\mu^2), G + \mu \frac{\partial \mathcal{L}^*}{\partial \alpha}(\alpha, G, e_J) + \mathcal{O}(\mu^2))$$

Remark: Note that from the second component of $\mathcal{S}_\mu$,

$$G_+ - G_- = \mu \frac{\partial \mathcal{L}^*}{\partial \alpha}(\alpha, G, e_J) + \mathcal{O}(\mu^2)$$

After checking that is o possible to define a scattering map, the following steps are necessary:

- Calculate the Melnikov Potential. (Fourier Series)
- To check that two different scattering maps ($\mathcal{L}$ is cosine-like + Poisson bracket)
- To construct a pseudo orbit and to apply shadow lemma.

The Melnikov Potential is

$$\mathcal{L}(\alpha, G, s; e_J) =$$
$$\int_{-\infty}^{\infty} \left\{ \frac{x_h^2(t)}{\left[4 + x_h^4(t) r(t+s)^2 + 4x_h^2(t) r(t+s)\cos(\alpha_h(t) - f(t+s))\right]^{1/2}} + \left(\frac{x_h^2(t)}{2}\right) r(t+s)\cos(\alpha_h(t) - f(t+s)) - \frac{x_h(t)}{2} \right\} dt$$

- $\mathcal{L}$ is computed via Fourier series on the angles α and s (This computation takes 30 pages!).
- $\mathcal{L}$ is an even function on the angles variables, then it has a Fourier Cosine series.

$$\mathcal{L}(\alpha, G, s; e_J) = L_{00} + 2\sum_{k\geq 1} L_{0k}\cos(k\alpha) + 2\sum_{q\geq 1}\sum_{k\geq 1} L_{qk}\cos(qs + k\alpha).$$

For $G > C$ for C large enough and $e_J G < c$ for c small enough

$$\mathcal{L}(\alpha, G, s; e_J) = \underbrace{\mathcal{L}_0(\alpha, G; e_J) + \mathcal{L}_1(\alpha, G, s; e_J)}_{\text{dominant part}} + \underbrace{\mathcal{L}_{\geq 2}(\alpha, G, s; e_J)}_{\text{exponentially small}}$$

$s \to \mathcal{L}$ is cosine-like if it has (only) two critical points, a maximum and a minimum.

As $\mathcal{L}_1(\alpha, G, s; e_J) = \mathcal{L}_1^*(\alpha, G; e_J) \cos(s - \alpha - \theta)$, the critical point are solution of

$$\frac{\partial \mathcal{L}}{\partial s}(\alpha, G, s; e_J) = -\mathcal{L}_1^*(\alpha, G; e_J) \sin(s - \alpha - \theta) + \frac{\partial \mathcal{L}_{\geq 2}}{\partial s}(\alpha, G, s; e_J) = 0$$

$$\Rightarrow \sin(s - \alpha - \theta) = \frac{1}{\mathcal{L}_1^*(\alpha, G; e_J)} \frac{\partial \mathcal{L}_{\geq 2}}{\partial s}(\alpha, G, s; e_J)$$

This equation has two solutions in $[-\pi, 3\pi/2]$ except in a small neighborhood.

There are two solutions are

$$s_+^* = \alpha + \theta + \phi \quad \text{and} \quad s_-^* = s_+^* + \pi,$$

for $\phi = \mathcal{O}(G^{-1/2}e^{-G^3/3})$
From these two critical points, it is defined two Reduced Poincaré functions:
$\mathcal{L}_\pm^*(\alpha, G, e_\mathrm{J}) = \mathcal{L}_0(\alpha, G, e_J) \pm +\mathcal{L}_1^*(\alpha, G, e_\mathrm{J}) + \xi_\pm(\alpha, G, e_\mathrm{J})$.

To check that the scattering maps associated to $\mathcal{L}_\pm^*$ are different, it is enough to check that the level curves of $\mathcal{L}_\pm^*$ are transversal, or,

$$\{\mathcal{L}_+^*, \mathcal{L}_-^*\} \neq 0.$$

- They are transversal in the region $G \geq C > 32$ amd $e_\mathrm{J} \leq c < 1/8$ except for three curves.

At any point in the plane (α, G), we choose the scattering map that $\frac{dG}{dt}$ is larger.
By applying this methodology is possible to construct a pseudo-orbit that presents a displacement with $\mathcal{O}(1)$.

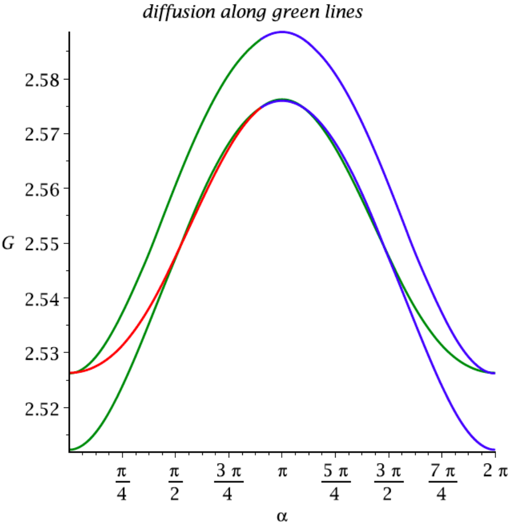


Fig. 3. Zone of diffusion: Level curves of $\mathcal{L}_+^*$ ($\mathcal{L}_-^*$) in blue (red) and diffusion trajectories in green (color figure online)

Thank you very much.

Tack så mycket.

Muchas gracias.

Muito obrigado.

- Deslshams, Kaloshin, de la Rosa and Seara. Global Instability in the Restricted Planar Elliptic Three Body Problem.*Communications in Mathematical Physics*.2019