Lecture 3: Scattering map

Arnold Diffusion and applications

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We consider a 2π-periodic in time perturbation of a pendulum and a rotor described by the non-autonomous Hamiltonian,

$$\begin{aligned} H_\varepsilon(p,q,I,\varphi,t) &= H_0(p,q,I) + \varepsilon h(p,q,I,\varphi,t;\varepsilon) \\ &= P_\pm(p,q) + \tfrac{1}{2}I^2 + \varepsilon h(p,q,I,\varphi,t;\varepsilon) \end{aligned} \tag{1}$$

where $(p,q,I,\varphi,t) \in (\mathbb{R}\times\mathbb{T})^2 \times \mathbb{T}$ and

$$P_\pm(p,q) = \pm\left(\frac{1}{2}p^2 + V(q)\right) \tag{2}$$

and $V(q)$ is a 2π-periodic function. We will refer to $P_\pm(p,q)$ as the *pendulum*.

Note. This model just comes from a normal form around a single resonance of a nearly integrable Hamiltonian. The perturbation is arbitrary.

Theorem (Delshams-Llave-Seara06)

Consider the Hamiltonian (1) where V and h are uniformly $\mathcal{C}^{r+2}$ for $r \geq r_0$, sufficiently large. Assume also that

H1 *The potential $V : \mathbb{T} \to \mathbb{R}$ has a unique global maximum at $q = 0$ which is non-degenerate. Denote by $(q_0(t), p_0(t))$ an orbit of the pendulum $P_\pm(p, q)$ homoclinic to $(0, 0)$.*

H2 *The Melnikov potential, associated to h (and to the homoclinic orbit (p_0, q_0)):*

$$\begin{aligned} \mathcal{L}(I, \varphi, s) \;=\; -\int_{-\infty}^{+\infty} & (h(p_0(\sigma), q_0(\sigma), I, \varphi + I\sigma, s + \sigma; 0) \\ & -h(0, 0, I, \varphi + I\sigma, s + \sigma; 0))d\sigma \end{aligned} \tag{3}$$

satisfies concrete non-degeneracy conditions.

H3 *The perturbation term h satisfies concrete non-degeneracy conditions.*

Then, there is $\varepsilon^* > 0$ such that for $0 < \varepsilon < \varepsilon^*$, and for any interval $[I_-^*, I_+^*]$, there exists a trajectory $\widetilde{x}(t)$ of the system (1) such that for some $T > 0$,

$$I(\widetilde{x}(0)) \leq I_-^*; \qquad I(\widetilde{x}(T)) \geq I_+^*.$$

Remark Arbitrary excursions in the I variable can also be realized.

Hypotheses **H1**, **H2** and **H3** are $\mathcal{C}^2$ generic, so, the following short version of the Theorem also holds:

Theorem (Delshams-Huguet09)

Consider the Hamiltonian (1) and assume that V and h are $\mathcal{C}^{r+2}$ functions which are $\mathcal{C}^2$ generic, with $r > r_0$, large enough. Then there is $\varepsilon^* > 0$ such that for $0 < |\varepsilon| < \varepsilon^*$ and for any interval $[I_-^*, I_+^*]$, there exists a trajectory $\widetilde{x}(t)$ of the system with Hamiltonian (1) such that for some $T > 0$

$$I(\widetilde{x}(0)) \leq I_-^*; \qquad I(\widetilde{x}(T)) \geq I_+^*.$$

Remark A (non optimal) value of r_0 which follows from our argument is $r_0 = 242$.

The main idea of the proof is to use the two (or more) dynamics on $\widetilde{\Lambda}$.

- Find a big invariant saddle object: a NHIM (normally hyperbolic invariant manifold: a global version of a center manifold) $\widetilde{\Lambda}$ with transverse associated stable and unstable manifolds along some homoclinic manifold Γ: $\mathcal{W}^u(\widetilde{\Lambda}) \pitchfork_\Gamma \mathcal{W}^s(\widetilde{\Lambda})$.
- Compute the invariant objects (typically tori $\mathcal{T}$) which may prevent instability for the inner dynamics of the NHIM.
- Compute an scattering map $S = S^\Gamma : H_- \subset \widetilde{\Lambda} \to H_+ \subset \widetilde{\Lambda}$ on the NHIM associated to Γ and consider it as an outer dynamics on the NHIM (a second dynamics on Γ).
- Check that $S(\mathcal{T}_{I_i}) \pitchfork \mathcal{T}_{I_{i+1}}$ for a sequence of tori $\{\mathcal{T}_{I_i}\}_{i=1}^N$ with $|I_N - I_1| = \mathcal{O}(1)$, and construct a transition chain of whiskered tori, i.e. $\mathcal{W}^u(\mathcal{T}_{I_i}) \pitchfork \mathcal{W}^s(\mathcal{T}_{I_{i+1}})$.
- Standard shadowing methods provide an orbit that follows closely the transition chain.

Consider a pendulum and a rotor plus a time periodic perturbation depending on two harmonics in the variables (φ, s):

$$H_\varepsilon(p, q, I, \varphi, t) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2} + \varepsilon h(q, \varphi, s) \tag{4}$$

$$\begin{gathered} h(q, \varphi, s) = f(q) g(\varphi, s), \\ f(q) = \cos q, \qquad g(\varphi, s) = a_1 \cos\varphi + a_2 \cos s. \end{gathered} \tag{5}$$

Theorem

Assume that $a_1 a_2 \neq 0$ in (4)-(5). Then, for any $I^* > 0$, there exists $\varepsilon^* = \varepsilon^*(I^*, a_1, a_2) > 0$ such that for any ε, $0 < \varepsilon < \varepsilon^*$, there exists a trajectory $(p(t), q(t), I(t), \varphi(t))$ such that for some $T > 0$

$$I(0) \leq -I^* < I^* \leq I(T).$$

We have two important dynamics associated to the system: the inner and the outer dynamics.

$$\widetilde{\Lambda} = \{(0, 0, I, \varphi, s); I \in [-I^*, I^*], (\varphi, s) \in \mathbb{T}^2\}.$$

is a 3D *Normally Hyperbolic Invariant Manifold* (NHIM) with associated 4D stable $W^s_\varepsilon(\widetilde{\Lambda})$ and unstable $W^u_\varepsilon(\widetilde{\Lambda})$ invariant manifolds.

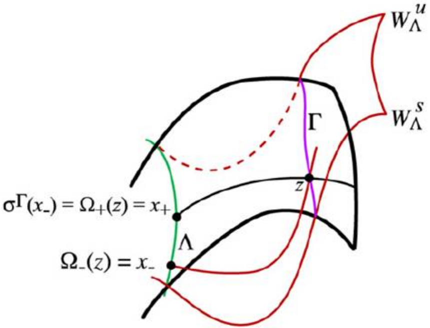
- The *inner dynamics* is the dynamics restricted to $\widetilde{\Lambda}$. (Inner map)
- The *outer dynamics* is the dynamics restricted to its invariant manifolds. (Scattering map)

Remark: for simplicity, in our case $\widetilde{\Lambda} = \widetilde{\Lambda}_\varepsilon$.

Let $\widetilde{\Lambda}$ be a NHIM with invariant manifolds intersecting transversally along a homoclinic manifold Γ. A scattering map is a map S defined by $S(\tilde{x}_-) = \tilde{x}_+$ if there exists $\tilde{z} \in \Gamma$ satisfying

$$|\phi_t^\varepsilon(\tilde{z}) - \phi_t^\varepsilon(\tilde{x}_\mp)| \longrightarrow 0 \text{ as } t \longrightarrow \mp\infty$$

that is, $W_\varepsilon^u(\tilde{x}_-)$ intersects transversally $W_\varepsilon^s(\tilde{x}_+)$ in $\tilde{z}$.



S is symplectic and exact (Delshams -de la Llave - Seara 2008) and takes the form:

$$S_\varepsilon(I,\varphi,s) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I,\theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I,\theta) + \mathcal{O}(\varepsilon^2), s \right),$$

where $\theta = \varphi - Is$ and $\mathcal{L}^*(I,\theta)$ is the Reduced Poincaré function, or more simply in the variables (I,θ):

$$\mathcal{S}_\varepsilon(I,\theta) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I,\theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I,\theta) + \mathcal{O}(\varepsilon^2) \right),$$

- The variable s remains fixed under S_ε: it plays the role of a parameter
- Up to first order in ε, $\mathcal{S}_\varepsilon$ is the $-\varepsilon$-time flow of the Hamiltonian $\mathcal{L}^*(I,\theta)$
- The scattering map jumps $\mathcal{O}(\varepsilon)$ distances along the level curves of $\mathcal{L}^*(I,\theta)$

To get a scattering map we search for homoclinic orbits to $\tilde{\Lambda}_\varepsilon$

Proposition

Given $(I,\varphi,s)\in[-I^*,I^*]\times\mathbb{T}^2$, assume that the real function

$$\tau\in\mathbb{R}\longmapsto\mathcal{L}(I,\varphi-I\tau,s-\tau)\in\mathbb{R}$$

has a non degenerate critical point $\tau^*=\tau(I,\varphi,s)$, where $\mathcal{L}(I,\varphi,s)=$

$$\int_{-\infty}^{+\infty}h(p_0(\sigma),q_0(\sigma),I,\varphi+I\sigma,s+\sigma;0)-h(0,0,I,\varphi+I\sigma,s+\sigma;0)d\sigma.$$

Then, for $0<|\varepsilon|$ small enough, there exists a transversal homoclinic point $\tilde{z}$ to $\widetilde{\Lambda}_\varepsilon$, which is ε-close to the point $\tilde{z}^*(I,\varphi,s)=(p_0(\tau^*),q_0(\tau^*),I,\varphi,s)\in W^0(\widetilde{\Lambda})$:

$$\tilde{z}=\tilde{z}(I,\varphi,s)=(p_0(\tau^*)+O(\varepsilon),q_0(\tau^*)+O(\varepsilon),I,\varphi,s)\in W^u(\widetilde{\Lambda}_\varepsilon)\pitchfork W^s(\widetilde{\Lambda}_\varepsilon).$$

In our model the perturbation is

$$h(p,q,I,\varphi,s) = \cos q\,(a_1\cos\varphi + a_2\cos s)$$

and the Melnikov potential becomes

$$\mathcal{L}(I,\varphi,s) = A_1(I)\cos(k_1\varphi + l_1 s) + A_2\cos(k_2\varphi + l_2 s),$$

where $A_1(I) = \dfrac{2\,\pi\,I\,a_1}{\sinh(\frac{I\pi}{2})}$ and $A_2 = \dfrac{2\pi\,a_2}{\sinh(\frac{\pi}{2})}$.

Definition

The Reduced Poincaré function is

$$\mathcal{L}^*(I,\theta) = \mathcal{L}(I,\varphi - I\,\tau^*(I,\varphi,s), s - \tau^*(I,\varphi,s)),$$

where $\theta = \varphi - I\,s$.

Plot of $\mathcal{L}(I, \varphi, s)$

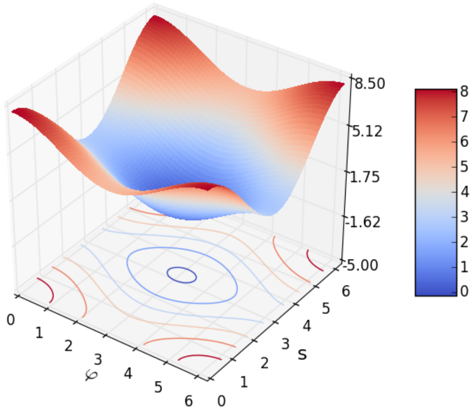


Figure: The Melnikov Potential, $\mu = a_1/a_2 = 0.6$, $I = 1$.

We look for τ^* such that $\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\,\tau^*, s - \tau^*) = 0$.

Different view-points for $\tau^* = \tau^*(I, \varphi, s)$

- Look for critical points of $\mathcal{L}$ on the straight line $R(I, \varphi, s) = \{(\varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$.
- Look for intersections between $R(I, \varphi, s) = \{(\varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$ and a crest which is a curve of equation

$$\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\tau, s - \tau)|_{\tau=0} = 0.$$

Definition - Crests (Delshams-Huguet 2011)

For each I, we call *crest* $\mathcal{C}(I)$ the set of curves in the variables (φ, s) of equation

$$I\frac{\partial\mathcal{L}}{\partial\varphi}(I,\varphi,s)+\frac{\partial\mathcal{L}}{\partial s}(I,\varphi,s)=0. \tag{6}$$

which in our case can be rewritten as

$$\mu\alpha(I)\sin\varphi+\sin s=0, \qquad \text{with } \alpha(I)=\frac{\sinh(\frac{\pi}{2})\,I^2}{\sinh(\frac{\pi I}{2})}, \quad \mu=\frac{a_{10}}{a_{01}}. \tag{7}$$

- For any I, the critical points of the Melnikov potential $\mathcal{L}(I,\cdot,\cdot)$ ($(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π): one maximum, one minimum point and two saddle points) always belong to the crest $\mathcal{C}(I)$.
- $\mathcal{L}^*(I,\theta)$ is nothing else but $\mathcal{L}$ evaluated on the crest $\mathcal{C}(I)$.
- $\theta=\varphi-Is$ is constant on the straight line $R(I,\varphi,s)$

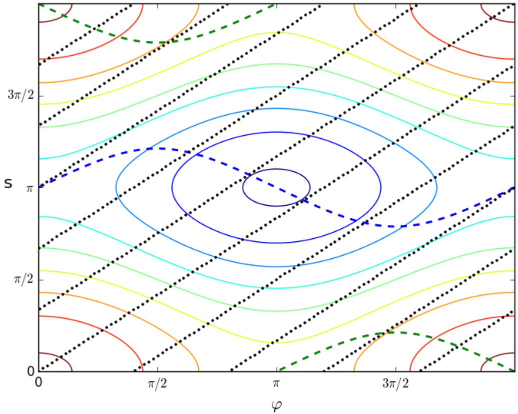


Figure: Level curves of $\mathcal{L}$ for $\mu = a_1/a_2 = 0.5$, $I = 1.2$.

Understanding the behavior of the crests

$\Downarrow$

Understanding the behavior of the Reduced Poincaré function

$\Downarrow$

Understanding the Scattering map

Definition: Highways

Highways are the level curves of $\mathcal{L}^*$ such that

$$\mathcal{L}^*(I,\theta) = \frac{2\pi a_1}{\sinh(\pi/2)}.$$

- The highways are "vertical" in the variables (φ, s)
- We always have a pair of highways. One goes up, the other goes down (this depends on the sign of $\mu = a_1/a_2$)
- The highways give rise to fast diffusing pseudo-orbits

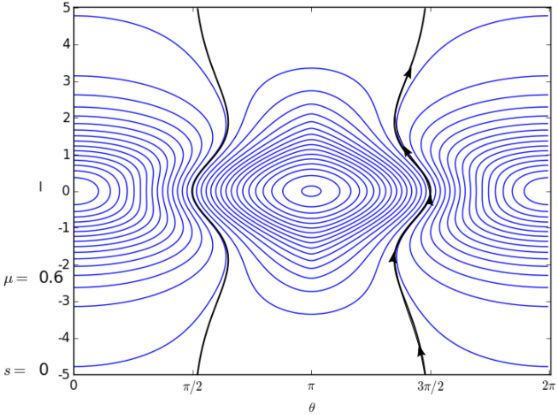
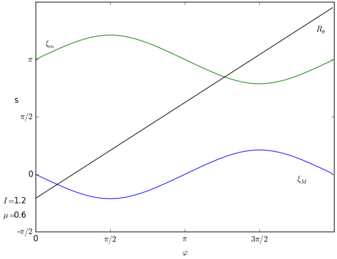


Figure: The scattering map jumps $\mathcal{O}(\varepsilon)$ distances along the level curves of $\mathcal{L}^*(I, \theta)$

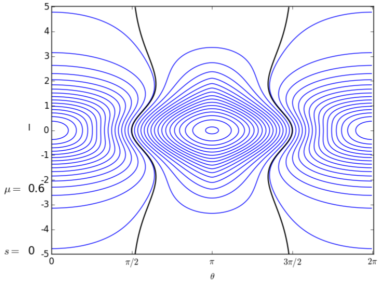
- For $|\mu\alpha(I)| < 1$, there are two crests $\mathcal{C}_{\mathrm{M,m}}(I)$ parameterized by:

$$
\begin{aligned}
s = \xi_M(I,\varphi) &= -\arcsin(\mu\alpha(I)\sin\varphi) && \mod 2\pi \\
\xi_m(I,\varphi) &= \arcsin(\mu\alpha(I)\sin\varphi) + \pi && \mod 2\pi
\end{aligned}
\tag{8}
$$

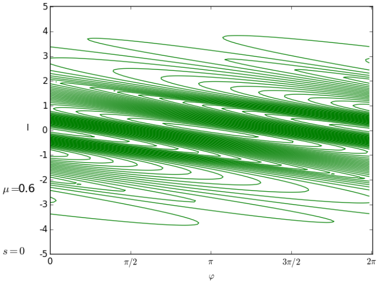


They are "horizontal" crests

- For each I, the line $R(I, \varphi, s)$ and the crest $\mathcal{C}_{\mathrm{M,m}}(I)$ have only one intersection point.
- The scattering map S_{M} associated to the intersections between $\mathcal{C}_{\mathrm{M}}(I)$ and $R(I, \varphi, s)$ is well defined for any $\varphi \in \mathbb{T}$. Analogously for S_{m}, changing M to m. In the variables $(I, \theta = \varphi - Is)$, both scattering maps $\mathcal{S}_{\mathrm{M}}$, $\mathcal{S}_{\mathrm{m}}$ are globally well defined.

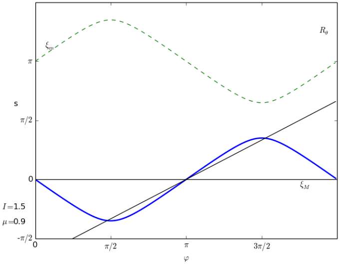


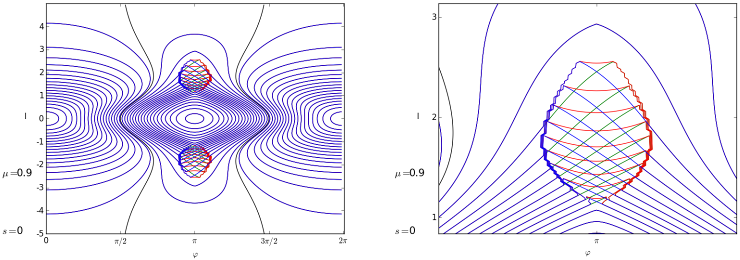
(a) Level curves of $\mathcal{L}^*_M(I, \theta)$



(b) Level curves of $\mathcal{L}^*_m(I, \theta)$

- There are tangencies between $\mathcal{C}_{\mathrm{M,m}}(I,\varphi)$ and $R(I,\varphi,s)$. For some value of (I,φ,s), there are 3 points in $R(I,\varphi,s)\cap\mathcal{C}_{\mathrm{M,m}}(I)$.
- This implies that there are 3 scattering maps associated to each crest with different domains.(Multiple Scattering maps)





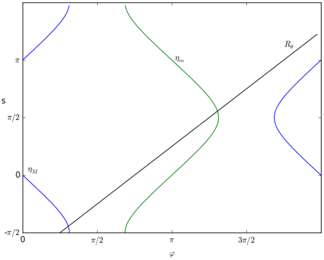
(c) The three types of level curves.

(d) Zoom where the scattering maps are different

Figure: Level curves of $\mathcal{L}^*_M(I,\theta)$, $\mathcal{L}^{*(1)}_M(I,\theta)$ and $\mathcal{L}^{*(2)}_M(I,\theta)$

- For some values of I, $|\mu\alpha(I)| > 1$, the two crests $\mathcal{C}_{\mathrm{M,m}}$ are parameterized by:

$$\begin{aligned} \varphi = \eta_M(I,s) &= -\arcsin(\mu\alpha(I)\sin s) \quad \mod 2\pi \\ \eta_m(I,s) &= \arcsin(\mu\alpha(I)\sin s) + \pi \quad \mod 2\pi \end{aligned} \tag{9}$$



They are "vertical" crests

For the values of I and when horizontal crests become vertical, it is not always possible to prolong in a continuous way the scattering maps, so the domain of the scattering map has to be restricted.

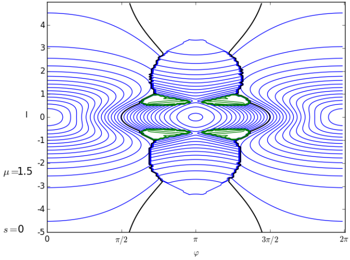


Figure: The level curves of $\mathcal{L}^*_{\mathrm{M}}(I,\theta)$, $\mu = 1.5$.

In green, the region where the scattering map S_{M} is not defined.

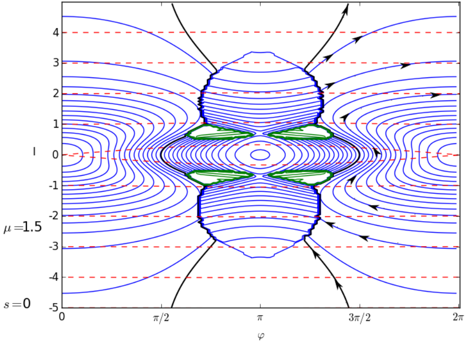


Figure: In red: Inner map, blue: Scattering map, black: Highways

An estimate of the total time of diffusion between $-I^*$ and I^*, along the highway, is

$$T_{\mathrm{d}} = \frac{T_{\mathrm{s}}}{\varepsilon}\left[2\log\left(\frac{C}{\varepsilon}\right) + \mathcal{O}(\varepsilon^b)\right], \text{ for } \varepsilon \to 0, \text{ where } 0 < b < 1,$$

with

$$T_{\mathrm{s}} = T_{\mathrm{s}}(I^*, a_1, a_2) = \int_0^{I^*} \frac{-\sinh(\pi I/2)}{\pi a_{10} I \sin\psi_{\mathrm{h}}(I)}\, dI,$$

where $\psi_{\mathrm{h}} = \theta - I\tau^*(I,\theta)$ is the parameterization of the highway $\mathcal{L}^*(I,\psi_{\mathrm{h}}) = A_2$, and

$$C = C(I^*, a_1, a_2) = 16\,|a_1|\left(1 + \frac{1.465}{\sqrt{1-\mu^2 A^2}}\right)$$

where $A = \max_{I\in[0,I^*]} \alpha(I)$, with $\alpha(I) = \frac{\sinh(\frac{\pi}{2})\, I^2}{\sinh(\frac{\pi I}{2})}$ and $\mu = a_1/a_2$.

Note: This estimate quantifies the general optimal diffusion estimate $\mathcal{O}\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ of [Berti-Biasco-Bolle 2003], [Cresson-Guillet 2003] and [Treschev 2004].

Thank you very much.

Tack så mycket.

Muchas gracias.

Muito obrigado.

- Deslshams and Schaefer. Arnold Diffusion for a Complete Family of Perturbations. *Regular and Chaotics Dynamics*.2017.