Lecture 2: Arnold's example, Nekhoroshev's estimates and Resonances

Arnold Diffusion and applications

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What is Global instability in Hamiltonian systems?

Assume a Hamiltonian system given by the Hamiltonian:

$$H_\varepsilon(\varphi, I) = h(I) + \varepsilon f(\varphi, I, t). \quad (1)$$

For $\varepsilon = 0$,

$$\dot{I} = \frac{\partial h}{\partial \varphi} = 0 \Rightarrow I = \text{constant}.$$

There exists a global instability in the action variable I if for a $\varepsilon \neq 0$, there exists an orbit of the system such that

$$\triangle I := |I(T) - I(0)| = \mathcal{O}(1).$$

This instability is also called Arnold diffusion.

Theorem (Arnold 63)

Suppose that the invariant torus N_{I_0} of the unperturbed system lies on the energy $\{H_0 = h\}$, the unperturbed system is isoenergetically nondegenerate at I_0:

$$det \begin{pmatrix} \frac{\partial^2 H_0}{\partial I^2}(I_0) & \omega(I_0) \\ \omega(I_0)^T & 0 \end{pmatrix} \neq 0, \qquad (2)$$

and the frequencies $\omega(I_0)$ are Diophantine. Then on the energy level $\{H = h\}$ of the perturbed system there is an invariant torus close to the original one. The frequencies on this torus are $\lambda\omega(I_0)$, where $\lambda = 1 + \mathcal{O}(\varepsilon)$.

For a Hamiltonian $H_\varepsilon(\varphi, I)$ with two degrees of freedom, that is, $(\varphi, I) \in \mathbb{T}^2 \times \mathcal{G} \subset \mathbb{R}^2$, we have the following theorem due to Arnold:

Theorem

In an isoenergetically non-degenerate system with two degrees of freedom, for all initial conditions, the action variables remain forever near their initial values.

1. Resonances
   - Single resonance normal form
   - Implementing one step
   - Hyperbolicity
   - Introducing $\mu = \sqrt{\varepsilon}$
   - Poincaré-Arnold-Melnikov
2. Arnold's example
   - Limitations
   - Arnold's conjecture
3. Nekhoroshev theorem
   - For the Arnold's example
4. Arnold diffusion
5. Research in Arnold diffusion

For a *nearly-integrable* Hamiltonian with n degrees of freedom

$$H_\varepsilon(\varphi, I) = h(I) + \varepsilon f(\varphi, I), \qquad (\varphi, I) \in \mathbb{T}^n \times \mathbb{R}^n,$$

we have studied the effect of a small perturbation on non-resonant (Diophantine) tori via KAM theorem.

Now, we are going to study on a neighborhood of a resonant torus.

Rewrite the H_ε as

$$H_\varepsilon(\varphi, I) = h(I) + \varepsilon f(\varphi, I), \qquad (\varphi, I) \in \mathbb{T}^{m+1} \times \mathbb{R}^{m+1},$$

where $m + 1 = n$.
Assume that the unperturbed Hamiltonian h has a single resonance at $I = I^*$.
Then, there exists a $k \in \mathbb{Z}^{m+1}$ such that $\langle \omega(I^*), k \rangle = 0$.
From the fact that it is single, any other k' satisfying such equality belongs to $\langle k \rangle$.

From a result of Abelian group theory, there is a change in the angle variables such that

$$\omega(I^*) = (0, \omega^*),$$

where $\omega^* \in \mathbb{R}^m$ and it is non-resonant.

By expanding in Taylor series, and assuming $I^* = 0$, the unperturbed Hamiltonian h can be written as

$$h(I) = \langle \omega(0), I \rangle + \frac{1}{2} \left\langle \partial_I^2 h(0) I, I \right\rangle + \mathrm{O}_3(I),$$

Replace $\varphi \to (q, \varphi)$ and $I \to (p, I)$ and

$$\partial_{p,I}^2 h(0) = \begin{pmatrix} \beta^2 & \lambda^\top \\ \lambda & \partial_I^2 h(0) \end{pmatrix},$$

where we have put $\beta^2 > 0$ in order to fix ideas, $\lambda \in \mathbb{R}^n$. We will assume $\beta = 1$; this can be achieved replacing p, I by p/β, I/β (changing in this way the time scale by a factor β), and rewriting ω^*/β, λ/β^2, $\partial_I^2 h(0)/\beta^2$ as ω^*, λ, Q respectively, and redefining also the function f.

Therefore, now, our Hamiltonian H_ε is

$$H_\varepsilon(q,p,\varphi,I) = h(p,I) + \varepsilon f(q,p,\varphi,I),$$

where

$$h(p,I) = \langle \omega^*, I\rangle + \frac{p^2}{2} + \langle \lambda, I\rangle\, p + \frac{1}{2}\langle QI, I\rangle + \mathrm{O}_3(p,I).$$

Now, it is performed a Normal form via Lie Series Method (For details see Delshams-Gutierrez 96):
Then we have to look for functions $S(q,\varphi)$ and $R(q,p,\varphi,I) = \mathrm{O}(p,I)$ such that

$$\{S,h\} + V + R = f, \tag{3}$$

where $V(q)$ is the periodic function obtained by averaging $f(q,0,\varphi,0)$ with respect to the angles φ:

$$V(q) = \overline{f(q,0,\cdot,0)} = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(q,0,\varphi,0)\mathrm{d}\varphi, \qquad q \in \mathbb{T}.$$

The time-1 symplectic flow Φ of the generating Hamiltonian εS leads to

$$
\begin{aligned}
H \circ \Phi &= H + \{H, \varepsilon S\} + \mathrm{O}\left(\varepsilon^2\right) \\
&= h + \varepsilon(V + R) + \mathrm{O}\left(\varepsilon^2\right) = H_0 + H_1,
\end{aligned}
$$

with

$$
\begin{aligned}
H_0(q, p, I; \varepsilon) &= \langle \omega^*, I \rangle + \frac{p^2}{2} + \varepsilon V(q) + \langle \lambda, I \rangle p + \frac{1}{2} \langle QI, I \rangle , \\
H_1(q, p, \varphi, I; \varepsilon) &= \varepsilon R(q, p, \varphi, I) + \mathrm{O}_3(p, I) + \mathrm{O}\left(\varepsilon^2\right) .
\end{aligned}
$$

with H_0 playing the role of the integrable Hamiltonian and H_1 being the perturbation.

We can assume that $V(q)$ has unique and non-degenerated maximum at q_0. Then, for $\varepsilon > 0$, the 1-degree-of-freedom Hamiltonian

$$P(q, p; \varepsilon) = \frac{p^2}{2} + \varepsilon V(q),$$

has a hyperbolic point in $(q_0, 0)$, with homoclinic separatrices.
Then, the Hamiltonian H_0 has whiskered tori with coincident whiskers associated to this saddle point.
For $\varepsilon < 0$ the same happens,but V has a minimum instead of a maximum.

The Lyapunov exponents of the saddle point of the "pendulum" P are $\pm\sqrt{\varepsilon}\alpha$, which tend to zero for $\varepsilon \to 0^+$.
To have fixed Lyapunov exponents, we can replace p, I by $\sqrt{\varepsilon}p$, $\sqrt{\varepsilon}I$.
The new system is still Hamiltonian if we divide the Hamiltonian by ε (making in this way a change of time scale by a factor $\sqrt{\varepsilon}$):

$$H_0 = \langle \omega, I \rangle + \frac{p^2}{2} + V(q) + \langle \lambda, I \rangle p + \frac{1}{2} \langle QI, I \rangle , \tag{4}$$

$$H_1 = R\left(x, \sqrt{\varepsilon}y, \varphi, \sqrt{\varepsilon}I\right) + \frac{1}{\varepsilon} \mathrm{O}_3\left(\sqrt{\varepsilon}y, \sqrt{\varepsilon}I\right) + \mathrm{O}\left(\varepsilon\right) = \mathrm{O}(\mu), \tag{5}$$

where

$$\omega = \frac{\omega^*}{\sqrt{\varepsilon}}, \qquad \mu = \sqrt{\varepsilon}.$$

This strategy of keeping $\varepsilon > 0$ fixed and letting $\mu \to 0$ was introduced by Poincaré in 1889 and followed in Arnold's example to avoid dealing with a singular perturbation problem.

Unfortunately, the *exponentially small splitting of separatrices* predicted by a direct application of the Poincaré-Arnold-Melnikov (PMA) method

$$\text{Splitting distance} = \varepsilon \text{ PMA prediction} + \mathrm{O}(\varepsilon\mu)$$

when the PMA prediction $= \mathrm{O}\left(\mathrm{e}^{-c/\varepsilon^{a}}\right)$ could then be justified only for μ exponentially small in ε.

Arnold's example

In 1964, V.I. Arnold proposed an example of a nearly-integrable Hamiltonian with $2 + 1/2$ degreees of freedom

$$H_\varepsilon(q, p, \varphi, I, t) = \frac{1}{2}\left(p^2 + I^2\right) + \varepsilon(\cos q - 1)\left(1 + \mu(\sin\varphi + \cos t)\right),$$

and asserted that given any $\delta, K > 0$, for any $0 < \mu \ll \varepsilon \ll 0$, there exists a trajectory of this Hamiltonian system such that

$$I(0) < \delta \text{ and } I(T) > K \qquad \text{for some time } T > 0.$$

Notice that this a global instability result for the variable I, since

$$\dot{I} = -\frac{\partial H_\varepsilon}{\partial \varphi} = -\varepsilon\mu(\cos q - 1)\cos\varphi$$

is zero for $\varepsilon = 0$, so I remains constant, whereas I can have a drift of finite size for *any* $\varepsilon > 0$ small enough.

Note this Hamiltonian H_ε can be written as

$$H_\varepsilon(q, p, \varphi, I, s) = h(q, p, I; \varepsilon) + \mu f(q, \phi, s; \varepsilon),$$

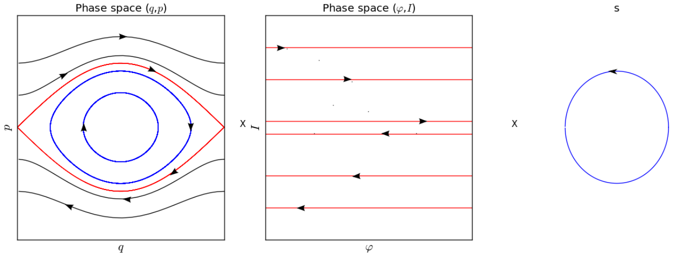
where h is the integrable Hamiltonian (Pendulum + rotor):

$$h(q, p, I) = \frac{p^2}{2} + \varepsilon(\cos q - 1) + \frac{I^2}{2}$$

And a perturbation in μ:

$$f(q, \varphi, s; \varepsilon) = \varepsilon \cos(q - 1)(\sin \varphi + s)$$

By assuming $\mu = 0$, we have the following phase space



Therefore, we have the presence of 2D tori

$$\mathcal{T}_\omega = \left\{(0,0,\varphi,\omega,s) : (\varphi,s) \in \mathbb{T}^2\right\}$$

And, associated to each tori above, we have the 3D invariant (homoclinic) manifold

$$W^s\mathcal{T}_\omega = W^u\mathcal{T}_\omega = \{(q_0(\sqrt{\varepsilon}\tau), \sqrt{\varepsilon}p_0(\sqrt{\varepsilon}\tau), I, \varphi, s) : \tau \in \mathbb{R}, (\varphi,s) \in \mathbb{T}^2\}$$

Arnold's example

Now, we consider $< \mu \ll \varepsilon$. By the special form of f, we have that $\mathcal{T}_\omega$ persist. But, their $W^u\mathcal{T}_\omega^\mu$ and $W^s\mathcal{T}_\omega^\mu$ do not coincide and intersect transversally.

More, for close enough tori, we have $W^u\mathcal{T}_{\omega_i}^\mu \pitchfork W^s\mathcal{T}_{\omega_{i+1}}^\mu$. Therefore, we can construct a sequence of tori $\mathcal{T}_{\omega_1}$,$\mathcal{T}_{\omega_2}$,...,$\mathcal{T}_{\omega_m}$, this sequence is called a transition chain. It possible to ensure by shadow result that there is a solution that travel along this chain.

The perturbation maintains fixed *all* the invariant tori $\mathcal{T}_\omega$. In general, there appear gaps around resonant tori (rational I) which prevent $W^s\widetilde{\mathcal{T}}^\varepsilon_{\omega_i} \pitchfork W^u\widetilde{\mathcal{T}}^\varepsilon_{\omega_{i+1}}$ because $\widetilde{\mathcal{T}}^\varepsilon_{\omega_i}$ and $\widetilde{\mathcal{T}}^\varepsilon_{I_{i+1}}$ are too far.

Arnold example only shows global instability along a single resonance, where the associated normal form is integrable, but does not deal with multiple resonances, where the normal form is not integrable.

Arnold conjectured Hamiltonian systems with three or more degrees of freedom are generically unstable.

Numerical experiments the the behavior of the diffusing orbits are similar to random walks.

Nekhoroshev proved, in 1977, that for generic systems diffusion happens exponentially slowly.

Theorem (Nekhoroshev theorem)

If the unperturbed Hamiltonian $h(I)$ is a steep[a] function, then there exist a, b, c such that in the perturbed Hamiltonian system for a sufficiently small perturbation we have

$$|I(t) - I(0)| \leq \varepsilon^{b} \qquad \text{for } |t| \leq (1/\varepsilon) \exp\left\{\left(c^{-1}/\varepsilon^{a}\right)\right\}.$$

[a] An analytic function is said to be steep if it has no stationary points and its restriction to any plane of any dimension has only isolated stationary points

The constants a, b and c are positive. a and b are called stability exponents.

Nekhoroshev theorem

If h is quasiconvex, that is, for any $I \in G$ and $v \in \mathbb{R}^n$,

$$Dh(I)v = 0 \text{ and } v \neq 0 \quad \Longrightarrow v^\top D^2 h(I) v \neq 0.$$

$a = b = \frac{1}{2n}$.

This theorem establishes Effective stability for all the trajectories of a steep nearly-integrable system

For the Anorld's example, we have that the unperturbed Hamiltonian can be written as

$$h(p, I, A) = \frac{1}{2}\left(p^2 + I^2\right) + A,$$

This Hamiltonian h is quasiconvex, then, by the Nekhoroshev theorem,

$$|(p, I, A)(t) - (p, I, A)(0)| \leq \varepsilon^{1/6} \qquad \text{for } |t| \leq (1/\varepsilon) \exp\left\{(\varepsilon_0/\varepsilon)^{1/6}\right\}.$$

Indeed, by [Pöschel93, Delshams-Gutiérrez96] for orbits close to the *single resonance* $p = 0$, using resonant normal forms, gives

$$a = b = 1/4.$$

Although Arnold diffusion is an expected phenomenon, from the obstacles generated by the KAM tori, the complexity of the dynamics close to resonances, and to be a slow process, the diffusion is not easy to detect theoretically neither numerically.

There are different approaches developed to detect this it: Variational (ex. Mather's set) and Geoemtrical (ex. scattering and separatrix map)

The Arnold diffusion is discussed in the possible scenarios [Chierchia and Gallovotti 94]:

- a priori stable Hamiltonian
  In this case, we have that the unperturbed Hamiltonian h has no hyperbolicity.
- a priori unstable Hamiltonian
  The unperturbed has a family of hyperbolic tori. This model is usually represented by a Pendulum + rotor system.
- a priori chaotic Hamiltonian
  The unperturbed Hamiltonian is not completely integrable. In general is related to geodesic flows.

Research in Arnold diffusion

Here, we have some examples of topics with active research

- Find explicit conditions, on h and εf, of generic type that the diffusion happens.(Arnold's conjecture)
- Different mechanisms of diffusion that cover the limitation of other ones.
- Example of systems that we can detect diffusion.
- Statistical properties of diffusing orbits and stochastic behavior of them
- Description of concrete diffusing orbits.
- Estimates of diffusing time and to find optimal diffusing orbits.

Thank you very much.

Tack så mycket.

Muchas gracias.

Muito obrigado.

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