Introduction, Nearly-Integrable Hamiltonian Systems and KAM theorem

Arnold Diffusion and applications

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Mini-course content:

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- Application to Celestial Mechanics

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Nowadays, this kind of problem can be described by a *nearly integrable* Hamiltonian system associated to

$$H_\varepsilon(\varphi, I) = H_0(I) + \varepsilon H_1(\varphi, I), \quad (\varphi, I) \in \mathbb{T}^n \times \mathbb{R}^n$$

with equations

$$\dot{\varphi} = \frac{\partial H_\varepsilon}{\partial I} \qquad \dot{I} = -\frac{\partial H_\varepsilon}{\partial \varphi},$$

and one wishes to understand the long term behavior of this system.

In general, the applications can be divided into two types of interest, stability and instability when $\varepsilon \neq 0$.

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- Instability:, the main question is to figure out how small forces produce large effects, or for instance, in a system in action-angle variables the existence of orbits whose actions change widely. Introduced by Arnold, this global instability is called Arnold diffusion.

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This instability is also called Arnold diffusion.

Nearly-integrable Hamiltonian

A nearly-integrable Hamiltonian in action-angle variables can be written in the form

$$H_\varepsilon(\varphi, I) = h(I) + \varepsilon f(\varphi, I), \tag{2}$$

where $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{T}^n$, $I = (I_1, \ldots, I_n) \in \mathcal{G} \subset \mathbb{R}^n$, ε is a small perturbation parameter, h is an integrable Hamiltonian. Then the Hamiltonian equations are

$$\dot{\varphi} = \omega(I) + \varepsilon \partial_I f(\varphi, I), \qquad \dot{I} = -\varepsilon \partial_\varphi f(\varphi, I).$$

The associated Hamiltonian equations for an unperturbed trajectory $(\varphi(t), I(t))$ are

$$\dot{\varphi} = \omega(I), \qquad \dot{I} = 0,$$

where $\omega = \partial_I h$. Hence the dynamics is very simple: every n-dimensional torus $I =$ constant is invariant, with linear flow

$$\varphi(t) = \varphi(0) + \omega(I)t,$$

and thus all trajectories are stable. The motion on a torus is called quasiperiodic, with associated frequencies given by the vector

$$\omega(I) = (\omega_1(I), \dots, \omega_n(I)).$$

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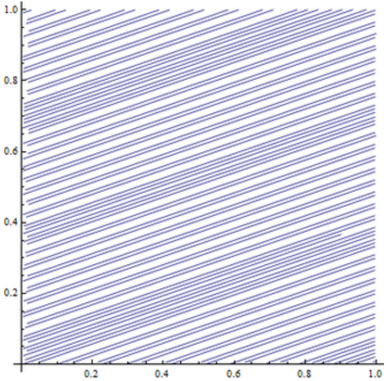
- Resonant otherwise.

A non-resonant torus is densely filled by any of its trajectories. On the other hand, a resonant torus is foliated into a family of lower dimensional tori.

Figure: Non-resonant 2D Torus



A frequency (of the unperturbed system) $\omega(I)$ is Diophantine if there exist positive constants c and γ such that

$$|\langle k, \omega(I)\rangle| \geq \frac{1}{c\,\|k\|^{\gamma}}$$

for any nonzero vector $k \in \mathbb{Z}^n$.

An invariant torus N_{I_0} is Diophantine if $\omega(I_0)$ is Diophantine.

Remark: A Diophantine torus is non-resonant.

In general lines, the KAM theory states that the most orbits lie on n-dimensional torus under a suitable non-degeneracy condition in ω and for a "small" perturbation εf.

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- Isoenergetic condition:

$$\det\begin{pmatrix} \frac{\partial\omega}{\partial I} & \omega \\ \omega^T & 0 \end{pmatrix} \neq 0.$$

  Non-resonant and resonant tori are dense on each energy level. The set of resonant tori has measure zero and the non-resonant has full measure.

These two conditions are independent:

Ex.

a) $h(I) = a_1 \log I_1 + a_2 \log I_2$, where $a_i \neq 0$ and $a_1 + a_2 = 0$. This Hamiltonian satisfies the standard degeneracy condition but it is isoenergetically degenerate.

For this Hamiltonian, the vector frequency is $\omega(I) = (a_1/I_1, a_2/I_2)$. Therefore,

$$\det\left(\frac{\partial\omega}{\partial I}\right) = \det\begin{pmatrix} -a_1/I_1^2 & 0 \\ 0 & -a_2/I_2^2 \end{pmatrix} = \frac{a_1 a_2}{I_1^2 I_2^2} \neq 0.$$

On the other hand, for the isoenergetic non-degeneracy,

$$\det\begin{pmatrix} \frac{\partial\omega}{\partial I} & \omega \\ \omega^T & 0 \end{pmatrix} = \det\begin{pmatrix} -a_1/I_1^2 & 0 & a_1/I_1 \\ 0 & -a_2/I_2^2 & a_2/I_2 \\ a_1/I_1 & a_2/I_2 & 0 \end{pmatrix} = \frac{a_1 a_2 (a_1 + a_2)}{I_1^2 I_2^2} = 0.$$

b) $h = I_1 + I_2^2/2$
This Hamiltonian is isoenergetically non-degenerated but (standard) degenerate.
For this Hamiltonian, $\omega(I) = (1, I_2)$. Therefore

$$\det \begin{pmatrix} \frac{\partial \omega}{\partial I} & \omega \\ \omega^T & 0 \end{pmatrix} = \det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & I_2 \\ 1 & I_2 & 0 \end{pmatrix} = -1 \neq 0.$$

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It is possible to state a KAM theorem for each non-degeneracy condition:

Theorem (Kolmogorov's theorem)

Suppose that the unperturbed system is non-degenerate at the point I_0:

$$\frac{\partial^2 h}{\partial I^2}(I_0) \neq 0,$$

and the torus N_{I_0} is Diophantine. Then, N_{I_0} survives the perturbation.It is just slightly deformed and as before carries quasiperiodic motions with the frequencies ω.

Note that a preserved invariant torus have the same vector frequency $\omega(I_0)$ that the unperturbed torus.

On the other hand, they are not, necessarily, at the same energy level.

Ex. Consider

$$H_\varepsilon(I) = h(I) + \varepsilon,$$

where $h(I) = a_1 \log I_1 + a_2 \log I_2$, where $a_i \neq 0$ and $a_1 + a_2 = 0$.

Suppose that $\omega(I^1) = c\omega(I^0)$, where $I = I^1$ is a torus for the perturbed Hamiltonian, $I = I^0$ is a torus for the unperturbed Hmamiltonian h, and $c > 0$.
In coordinates, this implies $I_1^1 = \frac{I_1^0}{c}$ and $I_2^1 = \frac{I_2^0}{c}$.

Therefore,

$$\begin{aligned} H_\varepsilon(I^1) = h(I^1) + \varepsilon &= a_1 \log I_1^1 + a_2 \log I_2^1 \\ &= a_1 \log(I_1^0/c) + a_2 \log(I_2^0/c) + \varepsilon \\ &= a_1 \log I_1^0 + a_2 \log I_2^0 - \log c\,(a_1 + a_2) + \varepsilon \\ &= h(I^0) + \varepsilon \neq h(I^0) \end{aligned}$$

Theorem (Arnold 63)

Suppose that the invariant torus N_{I_0} of the unperturbed system lies on the energy $\{H_0 = h\}$, the unperturbed system is isoenergetically nondegenerate at I_0:

$$det \begin{pmatrix} \frac{\partial^2 H_0}{\partial I^2}(I_0) & \omega(I_0) \\ \omega(I_0)^T & 0 \end{pmatrix} \neq 0, \qquad (3)$$

and the frequencies $\omega(I_0)$ are Diophantine. Then on the energy level $\{H = h\}$ of the perturbed system there is an invariant torus close to the original one. The frequencies on this torus are $\lambda\omega(I_0)$, where $\lambda = 1 + \mathcal{O}(\varepsilon)$.

Note that condition (3) is equivalent that $\omega \neq 0$ and

$$\frac{\partial \omega}{\partial I}(I_0)v + \lambda\omega(I_0) \neq 0, \quad \forall v \in \langle w(I_0)\rangle^{\perp} \setminus \{0\}, \forall \lambda \in \mathbb{R}.$$

This can be interpreted as traversality between the level energy $h^{-1}(I_0)$ and $\omega(I_0) \cdot v = 0$.

In this case, the value of h at the unperturbed torus is the same for H_ε at the perturbed torus. But the frequency is not preserved.

Ex.

$$H_\varepsilon(I) = h(I) + \varepsilon I_1,$$

where $h = I_1 + \frac{1}{2}\sum_{i=2}^{n} I_i^2$.

We have that the frequency for the unperturbed system for a torus $I = I^0$ is $\omega(I^0) = (1, I_2^0, \ldots, I_n^0)$. In the perturbed case, torus $I = I^0$ has frequency $\omega'(I^0) = (1 + \varepsilon, I_2^0, \ldots, I_n^0)$.

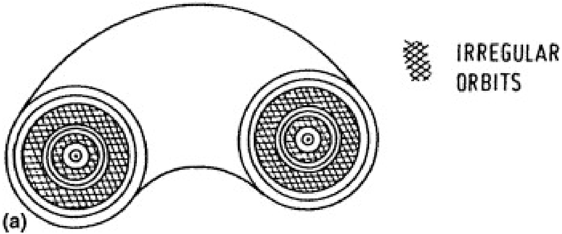
Then, $\omega(I^0) \neq \omega'(I^0)$. Note that $\omega(I^1) \neq \omega'(I^0)$ for any I^1, I^0.

For a Hamiltonian $H_\varepsilon(\varphi, I)$ with two degrees of freedom, that is, $(\varphi, I) \in \mathbb{T}^2 \times \mathcal{G} \subset \mathbb{R}^2$, we have the following theorem due to Arnold:

Theorem

In an isoenergetically non-degenerate system with two degrees of freedom, for all initial conditions, the action variables remain forever near their initial values.

Note that its phase space is four-dimensional. And, therefore energy levels are three dimensional. Therefore a 2D torus on an energy level separates such 3D is two. This implies that solutions between two torus are confined by them.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ORBITS
(a)

Note that this implies that in such case the diffusion is not possible.

This result is not true for the standard non-degeneracy condition.
Ex. Consider the Hamiltonian

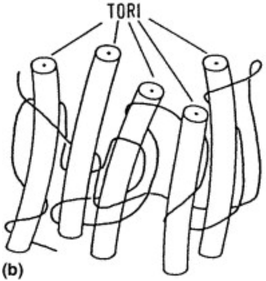
$$H_\varepsilon(\varphi, I) = \frac{\left(I_1^2 - I_2^2\right)}{2} + \varepsilon \sin\left(\varphi_1 - \varphi_2\right)$$

The frequency vector is $\omega(I) = (I_1, -I_2)$ for the unperturbed case, then

$$\det\left(\frac{\partial \omega}{\partial I}\right) = \det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \neq 0.$$

Note that $I_1 = -\varepsilon t$, $I_2 = \varepsilon t$, $\varphi_1 = -\varepsilon t^2/2$, and $\varphi_2 = -\varepsilon t^2/2$ is a solution. And it is easy to check that $|I(T) - 0| = \mathcal{O}(1)$ for some T.

For more degrees of freedom, such stability cannot be ensured under any non-degeneracy condition.
In particular, for 6 degrees of freedom, a 3D KAM invariant tori do not separate the 5D energy level as before, there can exist irregular orbits 'traveling' between tori.



Anyway, the isoenergetically non-degeneracy seems to be a stronger barrier to Arnold diffusion than others.

Other reason for the interest in this kind of non-degeneracy is the fact that it can eventually be applied to periodic non-autonomous Hamiltonian. The basic idea is to add an extra action variable conjugated to the time.

Note that we did not comment what happens for resonant tori. They are lower dimensional tori with stable and unstable manifolds. They can be responsible for generating hyperbolicity in the system that is useful for the construction of transition chains. Due to this unstable and stable manifold, they are called whiskered tori.

Thank you very much.

Tack så mycket.

Muchas gracias.

Muito obrigado.

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