Global instability in Hamiltonian systems

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The system

We consider the following *a priori unstable* Hamiltonian with $2 + \frac{1}{2}$ degrees of freedom with 2π-periodic time dependence:

$$H_\varepsilon(p, q, I, \varphi, s) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2} + \varepsilon \cos q \left(a_{00} + a_{10}\cos\varphi + a_{01}\cos s\right),$$

where p, $I \in \mathbb{R}$, q, φ, $s \in \mathbb{T}$ and ε is small enough.

In the unperturbed case, that is, $\varepsilon = 0$, the Hamiltonian H_0 is integrable (represents the standard pendulum plus a rotor):

$$H_0(p, q, I, \varphi, s) = \frac{p^2}{2} + \cos q - 1 + \frac{I^2}{2},$$

with associated equations:

$$\begin{aligned} \dot{q} &= \frac{\partial H_0}{\partial p} = p \qquad \dot{p} = -\frac{\partial H_0}{\partial q} = \sin q \\ \dot{\varphi} &= \frac{\partial H_0}{\partial I} = I \qquad \dot{I} = -\frac{\partial H_0}{\partial \varphi} = 0. \\ \dot{s} &= 1. \end{aligned} \tag{1}$$

and associated flow

$$\phi_t(p, q, I, \varphi, s) = (p(t), q(t), I, It + \varphi, t + s)\,.$$

I is constant.

Arnold diffusion

We have the following result:

Theorem

Consider a Hamiltonian of the form $H_\varepsilon(p,q,I,\varphi,t) = \frac{p^2}{2} + \cos q - 1 + \frac{I^2}{2} + \varepsilon\, f(q)\, g(\varphi,t)$, *where* $f(q) = \cos q$ *and* $g(\varphi,t) = a_{00} + a_{10}\,\cos\varphi + a_{01}\,\cos t$. *Assume that*

$$a_{10}\, a_{01} \neq 0$$

Then, for any $I^* > 0$, *there exists* $0 < \varepsilon^* = \varepsilon^*(I^*) << 1$ *such that for any* ε, $0 < \varepsilon < \varepsilon^*$, *there exists a trajectory* $(p(t), q(t), I(t), \varphi(t))$ *such that for some* $T > 0$

$$I(0) \leq -I^* < I^* \leq I(T).$$

We consider $\triangle I = \mathcal{O}(1)$, at least. This is an example of Arnold diffusion.

What makes this happen?

We have two important dynamics associated to the system: the inner and the outer dynamics.

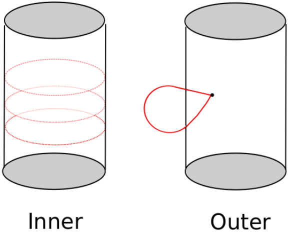
$$\widetilde{\Lambda} = \{\tau_I^0\}_{I\in[-I^*,I^*]} = \{(0,0,I,\varphi,s); I\in[-I^*,I^*], (\varphi,s)\in\mathbb{T}^2\}.$$

is a *Normally Hyperbolic Invariant Manifold* (NHIM)

- The *inner* is the dynamics restricted to $\widetilde{\Lambda}$. (Inner map)
- The *outer* is the dynamics restricted to its invariant manifolds. (Scattering map)

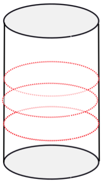
Inner and outer dynamics

The unperturbed case, $\varepsilon = 0$

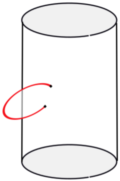


- Stable and unstable manifolds are *coincident.*
- The outer dynamics is the identity.

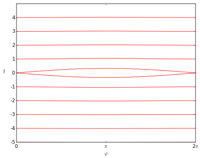
The perturbed case, $\varepsilon \neq 0$:



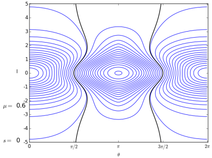
Inner



Outer



(a) Inner



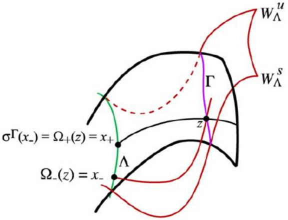
(b) Outer

- Stable and unstable manifolds, in general, are not coincident.
- The outer dynamics ensures the growth of I, that is, the Arnold diffusion.

Let $\widetilde{\Lambda}$ be a NHIM with invariant manifolds intersecting transversally along a homoclinic manifold Γ. A scattering map is a map S defined by $S(\tilde{x}_-) = \tilde{x}_+$ if there exists $\tilde{z} \in \Gamma$ satisfying

$$|\phi_t^\varepsilon(\tilde{z}) - \phi_t^\varepsilon(\tilde{x}_-)| \longrightarrow 0 \text{ as } t \longrightarrow -\infty$$
$$|\phi_t^\varepsilon(\tilde{z}) - \phi_t^\varepsilon(\tilde{x}_+)| \longrightarrow 0 \text{ as } t \longrightarrow +\infty,$$

that is, $W_\varepsilon^u(\tilde{x}_-)$ intersects transversally $W_\varepsilon^s(\tilde{x}_+)$ in $\tilde{z}$.



$S(I, \varphi, s)$ is symplectic and exact (Delshams -de la Llave - Seara 2000), this implies that S takes the form:

$$S_\varepsilon(I, \varphi, s) = \left(I + \varepsilon \frac{\partial L^*}{\partial \varphi}(I, \varphi, s) + \mathcal{O}(\varepsilon^2), \varphi - \varepsilon \frac{\partial L^*}{\partial I}(I, \varphi, s) + \mathcal{O}(\varepsilon^2), s \right),$$

or simply

$$\mathcal{S}_\varepsilon(I, \theta) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I, \theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I, \theta) + \mathcal{O}(\varepsilon^2) \right),$$

where $\theta = \varphi - Is$ and $\mathcal{L}^*(I, \theta)$ is the Reduced Poincaré function.

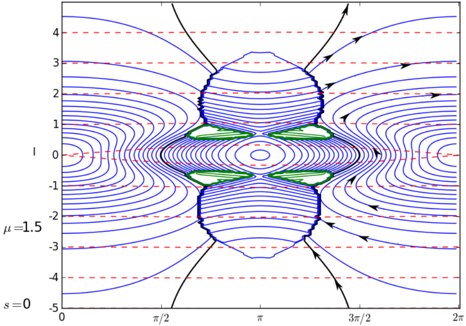
So, our focus will be the level curves of $\mathcal{L}^*(I, \theta)$.

Remark: The variable s remains fixed under the action of the Scattering map, or plays the role of a parameter.

Effectively, how does it ensure the Arnold diffusion?

Basically, we ensure the Arnold diffusion performing the following scheme:

- To construct a composition of Scattering map and Inner map. This composition is called a *pseudo-orbit*.
- To use previous results about Shadowing (Gidea-de la Llave - Seara 2014) for ensuring the existence of a real orbit close to our pseudo-orbit.



Is there the BEST pseudo-orbit?

Recall:

- Our perturbation is $\varepsilon \cos q \,(a_{00} + a_{10} \cos\varphi + a_{01} \cos s)$.
- the only hypothesis about it is $a_{10}a_{01} \neq 0$.

We have special curves, we called them Highways. In concrete, they are the level curves of $\mathcal{L}^*$ such that

$$\mathcal{L}^*(I,\theta) = 4a_{00} + \frac{2\pi a_{01}}{\sinh(\pi/2)}.$$

Why are they special? Because highways are "vertical"

We define $\mu = \frac{a_{10}}{a_{01}}$. Highways are defined in the following regions in the action I:

- for $|\mu| < 0.625$: $I \in (-\infty, +\infty)$
- for $0.625 \leq |\mu| \leq 1$: $(-\infty, -I_{++}) \cup (-I_+, I_+) \cup (I_{++}, +\infty)$, where

$$I_+ = \min\left\{I > 0 : \frac{I^3 \sinh(\pi/2)}{\sinh(I\pi/2)} = \frac{1}{|\mu|}\right\}$$

  and

$$I_{++} = \max\left\{I > 0 : \frac{I^3 \sinh(\pi/2)}{\sinh(I\pi/2)} = \frac{1}{|\mu|}\right\}$$

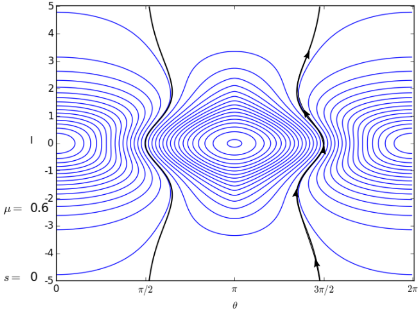
- for $|\mu| \geq 1$:$(-\infty, -I_{++}) \cup (-I_+, I_+) \cup (I_{++}, +\infty)$, where

$$I_+ = \min\left\{I > 0 : \frac{I^2 \sinh(\pi/2)}{\sinh(I\pi/2)} = \frac{1}{|\mu|}\right\}$$

  and

$$I_{++} = \max\left\{I > 0 : \frac{I^3 \sinh(\pi/2)}{\sinh(I\pi/2)} = \frac{1}{|\mu|}\right\}.$$

- We always have a "pair" of highways. One goes up, the other goes down (this depends on signal of μ.)
- It is easy to construct pseudo-orbits where highways are defined.



What is the Reduced Poincaré function?

Note that for scattering maps we have to look for homoclinic points. We will use the Melnikov Potential:

Proposition

Given $(I, \varphi, s) \in [-I^*, I^*] \times \mathbb{T}^2$, assume that the real function

$$\tau \in \mathbb{R} \longmapsto \mathcal{L}(I, \varphi - I\,\tau, s - \tau) \in \mathbb{R}$$

has a non degenerate critical point $\tau^* = \tau(I, \varphi, s)$, where $\mathcal{L}(I, \varphi, s) =$

$$\int_{-\infty}^{+\infty} h(p_0(\sigma), q_0(\sigma), I, \varphi + I\sigma, s + \sigma; 0) - h(0, 0, I, \varphi + I\sigma, s + \sigma; 0) d\sigma.$$

Then, for $0 < |\varepsilon|$ small enough, there exists a transversal homoclinic point $\tilde{z}$ to $\widetilde{\Lambda}_\varepsilon$, which is ε-close to the point $\tilde{z}^*(I, \varphi, s) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) \in W^0(\widetilde{\Lambda})$:

$$\tilde{z} = \tilde{z}(I, \varphi, s) = (p_0(\tau^*) + O(\varepsilon), q_0(\tau^*) + O(\varepsilon), I, \varphi, s) \in W^u(\widetilde{\Lambda}_\varepsilon) \pitchfork W^s(\widetilde{\Lambda}_\varepsilon).$$

In our model, $h(p, q, I, \varphi, s) = \cos q\,(a_0 0 + a_{01} \cos\varphi + a_{01} \cos s)$.

- $\mathcal{L}$ is the Melnikov potential.
- In our case

$$\mathcal{L}(I,\varphi,s) = A_{00} + A_{10}(I)\cos\varphi + A_{01}\cos s, \qquad (2)$$

where $A_{00} = 4\,a_{00}$, $A_{10}(I) = \dfrac{2\,\pi\,I\,a_{10}}{\sinh(\frac{I\,\pi}{2})}$ and $A_{01} = \dfrac{2\,\pi\,a_{01}}{\sinh(\frac{\pi}{2})}$.

- We look for τ^* such that

$$\frac{\partial\mathcal{L}}{\partial\tau}(I,\varphi - I\,\tau^*, s - \tau^*) = 0.$$

In our case, we look for τ^* such that:

$$I\,A_{10}(I)\sin(\varphi - I\,\tau^*) + A_{10}\sin(s - \tau^*) = 0. \qquad (3)$$

The Reduced Poincaré function

We define the Reduced Poincaré functions as

$$\mathcal{L}^*(I,\theta) = \mathcal{L}(I, \varphi - I\,\tau^*(I,\varphi,s), s - \tau^*(I,\varphi,s)),$$

where $\theta = \varphi - I\,s$.

- It is evaluated on the critical points of $\mathcal{L}$ on the straight line $R(I,\varphi,s) = \{(\varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$. Besides θ is constant on the straight line.
- From another view-point, it is evaluated on the intersection between $R(I,\varphi,s) = \{(\varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$ and the curve of equation

$$I\,A_{10}(I)\sin\varphi + A_{01}\sin s = 0.$$

.

Crests

Definition - Crests (Delshams-Huguet 2011)

For each I, we call *crests* the pair (φ, s) such that $\tau^* = 0$ satisfies the equation (3), that is,

$$I\, A_{10}(I) \sin\varphi + A_{01} \sin s = 0. \tag{4}$$

For the computation of the reduced Poincaré function, we have to study this equation. We can rewrite it as

$$\mu\alpha(I)\, \sin\varphi + \sin s = 0, \tag{5}$$

where

$$\alpha(I) = \frac{\sinh(\frac{\pi}{2})\, I^2}{\sinh(\frac{\pi I}{2})} \tag{6}$$

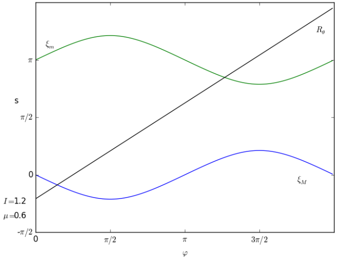
and

$$\mu = \frac{a_{10}}{a_{01}}. \tag{7}$$

$0 < |\mu| < 0.97$

- $|\mu\alpha(I)| < 1$, there are two crests $\mathcal{C}_{\mathsf{M,m}}(I)$ parameterized by:

$$s = \xi_M(I,\varphi) = -\arcsin(\alpha(I,\mu)\sin\varphi) \mod 2\pi \quad (8)$$
$$\xi_m(I,\varphi) = \arcsin(\alpha(I,\mu)\sin\varphi) + \pi \mod 2\pi$$



They are the horizontal crests

$0 < |\mu| < 0.625$

- For each I, the line $R(I, \varphi, s)$ and the crest $\mathcal{C}_{\mathrm{M,m}}(I)$ have only one intersection point.
- The intersection is always transversal.
- We have well defined S_M and S_m, where S_{M} is the scattering map associated to the intersections between $\mathcal{C}_{\mathrm{M}}(I)$ and $R(I, \varphi, s)$ and S_{m} is the scattering map associated to the intersection between $\mathcal{C}_{\mathrm{m}}(I)$ and $R(I, \varphi, s)$.

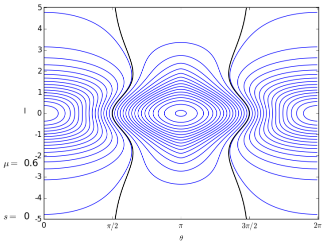
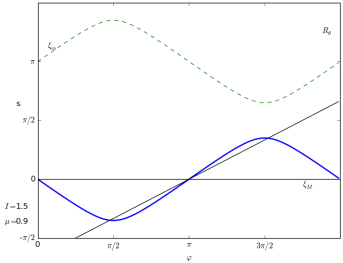


Figura: Level curve of $\mathcal{L}^*$ associated to $\mathcal{C}_{\mathrm{M}}(I)$.

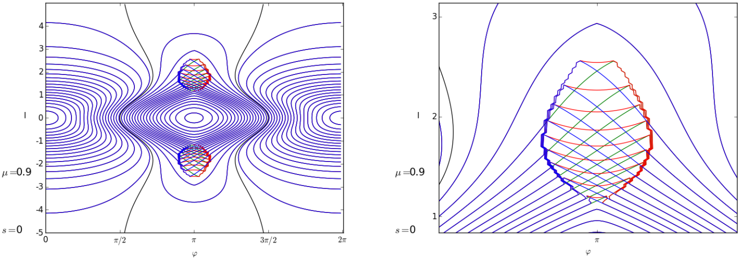
- The equations of the crests are the same.
- There are tangencies between $\mathcal{C}_{\mathsf{M,m}}(I,\varphi)$ and $R(I,\varphi,s)$. If $\theta \neq \pi$, the tangency happens for two angles. In this case, for some value of (φ, s), there are 3 points in $R(I,\varphi,s) \cap \mathcal{C}_{\mathsf{M,m}}(I)$.
- The item above implies that there are three scattering maps associated to each crest. In this case we have Multiple Scattering maps.



We define as tangency locus the set

$$\left\{ (I,\theta); \frac{\partial \xi}{\partial \varphi}(I,\varphi) = \frac{1}{I} \right\}.$$

- Out of the delimited region by the tangency locus: Scattering maps are equal.
- In this region, they are different.

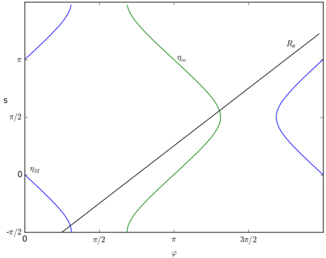


(a) The three types of level curves. (b) Zoom around the tangency locus

$|\mu| > 0.97$

- For some values of I, $|\mu\alpha(I)| > 1$, the two crests $\mathcal{C}_{\mathsf{M,m}}$ are parameterized by:

$$\begin{aligned} \varphi = \eta_M(I,s) &= -\arcsin(\alpha(I,\mu)\sin s) && \mod 2\pi \quad (9) \\ \eta_m(I,s) &= \arcsin(\alpha(I,\mu)\sin s) + \pi && \mod 2\pi \end{aligned}$$



They are the vertical crests

As this happens for some values of I and when it happens, we can look this crests locally as the horizontal crests, we restrict the domain of the Scattering map.

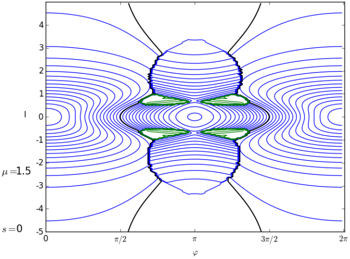


Figura: In green, the region where the scattering map is not defined.

In this talk we have just displayed Scattering maps with $s = 0$. But if we change its value in the formula

$$S_\varepsilon(I,\varphi,s) = \left(I + \varepsilon \frac{\partial L^*}{\partial \varphi}(I,\varphi,s) + \mathcal{O}(\varepsilon^2), \varphi - \varepsilon \frac{\partial L^*}{\partial I}(I,\varphi,s) + \mathcal{O}(\varepsilon^2), s \right),$$

we have more options for the diffusion, that is, the pseudo-orbit.

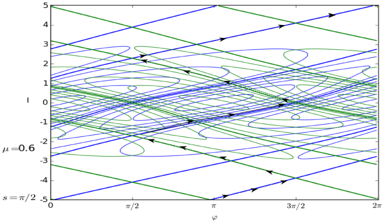


Figura: The level curves of the Reduced Poincaré function associated to $\mathcal{C}_{\mathsf{M}}(I)$ in blue, and associated to $\mathcal{C}_{\mathsf{m}}(I)$ in green, $s = \pi/2$.

Time of diffusion

An estimate of the total time of diffusion between I_0 and I_{f} along the highways is

$$T_d \sim N_{\mathsf{s}} T_{\mathsf{h}},$$

where

- $T_{\mathsf{h}} = \log\left(\frac{4(|a_{00}|+|a_{10}|+|a_{01}|)}{\varepsilon}\right)$ is the time along the homoclinic invariant manifold of $\widetilde{\Lambda}$
- $N_{\mathsf{s}} = T_{\mathsf{s}}/\varepsilon$ is the number of iterates of the scattering map along the highway and
- $T_s = \int_{I_0}^{I_f} \frac{-\sinh(I\pi/2)}{2\pi I a_{10}\sin\psi_{\mathsf{h}}(I)} dI$, where $\psi_{\mathsf{h}} = \theta - I\tau^*(I,\theta)$ is a parametrization of the highway.

This estimate agrees with the optimal estimate of (Berti-Biasco-Bolle 2003) and (Treschev 2004), that is, a time of the order $\mathcal{O}(\varepsilon^{-1}\log\varepsilon^{-1})$.

Thank you for your attention.

A short bibliography

- A geometric mechanism of diffusion: Rigorous verification in a priori unstable Hamiltonian systems - (Delshams - Huguet 2011)
- A Geometric Approach to the Existence of Orbits with Unbounded Energy in Generic Periodic Perturbations by a Potential of Generic Geodesic Flows of $\mathbb{T}^2$ (Delshams - de la Llave -Seara 2000) (for Scattering maps)
- A general mechanism of diffusion in Hamiltonian systems: Qualitative results (Gidea - de la Llave - Seara 2014) (for Shadowing)
- Drift in phase space: a new variational mechanism with optimal diffusion time (Berti - Biasco - Bolle 2003)
- Evolution of slow variables in a priori unstable Hamiltonian systems (Treschev - 2004)