Arnold diffusion for a Hamiltonian with $3 + 1/2$ degrees of freedom

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Global instability

What is Global instability in Hamiltonian systems?

Assume a Hamiltonian system given by the Hamiltonian:

$$H(q,p,I,\varphi) = h_0(q,p,I) + \varepsilon h_1(q,p,I,\varphi,t). \tag{1}$$

For $\varepsilon = 0$,

$$\dot{I} = \frac{\partial h_0}{\partial \varphi} = 0 \Rightarrow I = \text{constant}. \tag{2}$$

There exists a global instability in the variable I if for a $\varepsilon \neq 0$, there exists an orbit of the system (1) such that

$$\triangle I := |I(T) - I(0)| = \mathcal{O}(1). \tag{3}$$

This instability is also called Arnold diffusion.

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In 1964, V.I. Arnold proposed an example of a nearly-integrable Hamiltonian with $2 + 1/2$ degrees of freedom

$$H(q, p, \varphi, I, t) = \frac{1}{2}\left(p^2 + I^2\right) + \varepsilon(\cos q - 1)\left(1 + \mu(\sin \varphi + \cos t)\right),$$

and asserted that given any $\delta, K > 0$, for any $0 < \mu \ll \varepsilon \ll 0$, there exists a trajectory of this Hamiltonian system such that

$$I(0) < \delta \text{ and } I(T) > K \qquad \text{for some time } T > 0.$$

Notice that this a global instability result for the variable I, since

$$\dot{I} = -\frac{\partial H}{\partial \varphi} = -\varepsilon\mu(\cos q - 1)\cos \varphi$$

is zero for $\varepsilon = 0$, so I remains constant, whereas I can have a drift of finite size for *any* $\varepsilon > 0$ small enough.

Arnold's Hamiltonian can be written as a nearly-integrable autonomous Hamiltonian with 3 degrees of freedom

$$H^*(q,p,\varphi,I,s,A) = \frac{1}{2}\left(p^2+I^2\right) + A + \varepsilon(\cos q - 1)\left(1+\mu(\sin\varphi + \cos s)\right),$$

which for $\varepsilon = 0$ is an integrable Hamiltonian $h(p,I,A) = \frac{1}{2}\left(p^2+I^2\right)+A$. Since h satisfies the (Arnold) *isoenergetic nondegeneracy*

$$\begin{vmatrix} D^2 h & Dh \\ Dh^\top & 0 \end{vmatrix} = -1 \neq 0$$

By the KAM theorem proven by Arnold in 1963, the 5D phase space of H is filled, up to a set of relative measure $\mathrm{O}(\sqrt{\varepsilon})$, with 3D-invariant tori $\mathcal{T}_\omega$ with Diophantine frequencies $\omega = (\omega_1,\omega_2,1)$:

$$|k_1\omega_1 + k_2\omega_2 + k_0| \geq \gamma/|k|^\tau \text{ for any } 0 \neq (k_1,k_2,k_0) \in \mathbb{Z},$$

where $\gamma = \mathrm{O}(\sqrt{\varepsilon})$, and $\tau \geq 2$.

Consider a pendulum and two s plus a time periodic perturbation depending on three harmonics in the variables $\varphi = (\varphi_1, \varphi_2)$ and s:

$$H_\varepsilon(p, q, I, \varphi, s) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{\Omega_1 I_1^2}{2} + \frac{\Omega_2 I_2^2}{2} + \varepsilon h(q, \varphi, s) \quad (4)$$

$$\begin{aligned} & h(q,\varphi,s) = f(q)g(\varphi,s), \\ f(q) = \cos q, \qquad & g(\varphi,s) = a_1 \cos\varphi_1 + a_2\cos\varphi_2 + a_3 \cos s. \end{aligned} \quad (5)$$

Theorem

Consider the Hamiltonian (4)+(5). Assume $a_1 a_2 a_3 \neq 0$ and $|a_1/a_3| + |a_2/a_3| < 0.625$. Then, for every $\delta < 1$ and $R > 0$ there exists $\varepsilon_0 > 0$ such that for every $0 < |\varepsilon| < \varepsilon_0$, given $|I_\pm| \leq R$, there exists an orbit $\tilde{x}(t)$ and $T > 0$, such that

$$|I(\tilde{x}(0)) - I_-| \leq \delta \quad \text{and} \quad |I(\tilde{x}(T)) - I_+| \leq \delta.$$

- To review the construction of scattering maps initiated in [Delshams-Llave-Seara00], designed to detect global instability.
- To play with the parameter $\mu_1 = a_1/a_3$ and $\mu_2 = a_2/a_3$ to show their influence in our mechanism.
- To present some diffusion results for this concrete model with $3 + 1/2$ degrees of freedom.

We deal with an a priori unstable Hamiltonian [Chierchia-Gallavotti94].

In the unperturbed case $\varepsilon = 0$, the Hamiltonian H_0 is integrable formed by the standard pendulum plus two rotors

$$H_0(p, q, I, \varphi, s) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{\Omega_1 I_1^2}{2} + \frac{\Omega_2 I_2^2}{2}.$$

$I = (I_1, I_2)$ is constant: $\quad \triangle I := |I(T) - I(0)| \equiv 0.$

For any $0 < \varepsilon \ll 1$, there is a finite drift in the action of the rotor I: $\triangle I = \mathcal{O}(1)$, so we have global instability.

In short, this is is also frequently called Arnold diffusion.

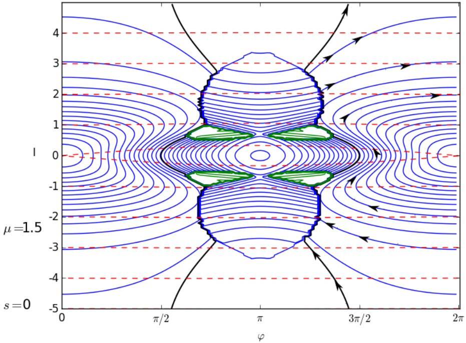
Basically, we ensure the Arnold diffusion performing the following scheme:

- To construct iterates under several Scattering maps and the Inner map, giving rise to diffusing pseudo-orbits.
- To use previous results about Shadowing [Fontich-Martín00], [Gidea-Llave-Seara14] for ensuring the existence of real orbits close to the pseudo-orbits.

As an illustration of our mechanics, we show an example for $2 + 1/2$ degrees of freedom:

$$H_\varepsilon(p, q, I, \varphi) = \pm \left(\frac{p^2}{2} + \cos q - 1 \right) + \frac{I^2}{2} + \varepsilon \cos q \left(\mu \cos \varphi + \cos s \right).$$

This case was studied in [Delshams - S. 2017].



We have two important dynamics associated to the system: the inner and the outer dynamics on a large invariant object $\widetilde{\Lambda}$.

$$\widetilde{\Lambda} = \{(0, 0, I, \varphi, s); I \in [-I^*, I^*]^2, (\varphi, s) \in \mathbb{T}^3\}.$$

is a 5D *Normally Hyperbolic Invariant Manifold* (NHIM) with associated 6D stable $W^{\mathrm{s}}_{\varepsilon}(\widetilde{\Lambda})$ and unstable $W^{\mathrm{u}}_{\varepsilon}(\widetilde{\Lambda})$ invariant manifolds.

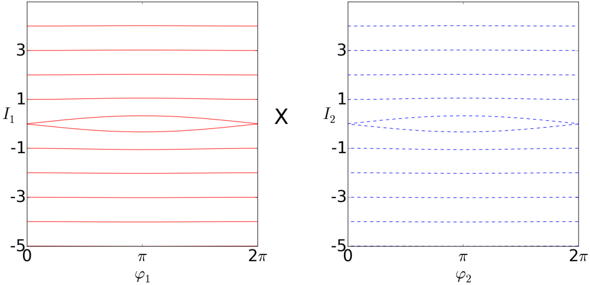
- The *inner dynamics* is the dynamics restricted to $\widetilde{\Lambda}$. (Inner map)
- The *outer dynamics* is the dynamics along the invariant manifolds of $\widetilde{\Lambda}$. (Scattering map)

Remark: Due to the form of the perturbation, $\widetilde{\Lambda} = \widetilde{\Lambda}_{\varepsilon}$ (not essential).

As we have $g(\varphi, s) = a_1 \cos\varphi_1 + a_2 \cos\varphi_2 + a_3 \cos s$, the inner dynamics is described by the Hamiltonian system with the Hamiltonian

$$K(I, \varphi, s) = \frac{\Omega_1 I_1^2}{2} + \frac{\Omega_2 I_2^2}{2} + \varepsilon\left(a_1 \cos\varphi_1 + a_2 \cos\varphi_2 + \cancel{a_3 \cos s}\right).$$

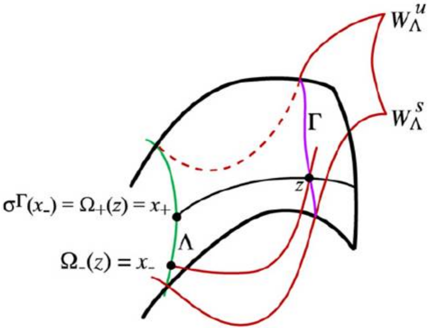
In this case the inner dynamics is integrable.



Let $\widetilde{\Lambda}$ be a NHIM with invariant manifolds intersecting transversally along a homoclinic manifold Γ. A scattering map is a map S defined by $S(\tilde{x}_-) = \tilde{x}_+$ if there exists $\tilde{z} \in \Gamma$ satisfying

$$|\phi_t^\varepsilon(\tilde{z}) - \phi_t^\varepsilon(\tilde{x}_\mp)| \longrightarrow 0 \text{ as } t \longrightarrow \mp\infty$$

that is, $W_\varepsilon^u(\tilde{x}_-)$ intersects transversally $W_\varepsilon^s(\tilde{x}_+)$ in $\tilde{z}$.



S is an exact symplectic map [Delshams-Llave-Seara08] and takes the form:

$$S_\varepsilon(I,\varphi,s) = \left(I + \varepsilon\,\frac{\partial \mathcal{L}^*}{\partial \theta}(I,\theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon\,\frac{\partial \mathcal{L}^*}{\partial I}(I,\theta) + \mathcal{O}(\varepsilon^2), s\right),$$

where $\theta = \varphi - Is$ and $\mathcal{L}^*(I,\theta)$ is the Reduced Poincaré function, or more simply in the variables (I,θ):

$$\mathcal{S}_\varepsilon(I,\theta) = \left(I + \varepsilon\,\frac{\partial \mathcal{L}^*}{\partial \theta}(I,\theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon\,\frac{\partial \mathcal{L}^*}{\partial I}(I,\theta) + \mathcal{O}(\varepsilon^2)\right),$$

- The variable s remains fixed under S_ε: it plays the role of a parameter
- Up to first order in ε, $\mathcal{S}_\varepsilon$ is the $-\varepsilon$-time flow of the Hamiltonian $\mathcal{L}^*(I,\theta)$
- The scattering map jumps $\mathcal{O}(\varepsilon)$ distances along the level curves of $\mathcal{L}^*(I,\theta)$

Now, we are going to construct the Reduced Poincaré function $\mathcal{L}^*$.

To get a scattering map we search for homoclinic orbits to $\tilde{\Lambda}_\varepsilon$

Proposition

Given $(I, \varphi, s) \in [-I^*, I^*]^2 \times \mathbb{T}^3$, assume that the real function

$$\tau \in \mathbb{R} \longmapsto \mathcal{L}(I, \varphi - I\tau, s - \tau) \in \mathbb{R}$$

has a non degenerate critical point $\tau^* = \tau(I, \varphi, s)$, where

$$\mathcal{L}(I, \varphi, s) = \int_{-\infty}^{+\infty} (\cos q_0(\sigma) - \cos 0)\, g(\varphi + I\sigma, s + \sigma; 0) d\sigma.$$

Then, for $0 < |\varepsilon|$ small enough, there exists a transversal homoclinic point $\tilde{z}$ to $\widetilde{\Lambda}_\varepsilon$, which is ε-close to the point $\tilde{z}^*(I, \varphi, s) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) \in W^0(\widetilde{\Lambda})$:

$$\tilde{z} = \tilde{z}(I, \varphi, s) = (p_0(\tau^*) + O(\varepsilon), q_0(\tau^*) + O(\varepsilon), I, \varphi, s) \in W^u(\widetilde{\Lambda}_\varepsilon) \pitchfork W^s(\widetilde{\Lambda}_\varepsilon).$$

In our model $q_0(t) = 4 \arctan \mathrm{e}^t$, $p_0(t) = 2/\cosh t$ is the separatrix for positive p of the standard pendulum $P(q,p) = p^2/2 + \cos q - 1$.
For our $g(\varphi, s) = a_1 \cos \varphi_1 + a_2 \cos \varphi_2 + a_3 \cos s$, the Melnikov potential becomes

$$\mathcal{L}(I, \varphi, s) = A_1(I_1) \cos \varphi_1 + A_2(I_2) \cos \varphi_2 + A_3 \cos s,$$

where $A_i(I_i) = \dfrac{2\,\pi\,\Omega_i I_i\, a_i}{\sinh\left(\frac{\Omega_i I_i\,\pi}{2}\right)}$, $i = \{1, 2\}$ and $A_3 = \dfrac{2\,\pi\, a_3}{\sinh\left(\frac{\pi}{2}\right)}$.

Finally, the function $\mathcal{L}^*(I,\theta)$ can be defined:

Definition

The Reduced Poincaré function is

$$\mathcal{L}^*(I,\theta) = \mathcal{L}(I, \varphi - I\,\tau^*(I,\varphi,s), s - \tau^*(I,\varphi,s)),$$

where $\theta = \varphi - I\,s$.

Therefore the definition of $\mathcal{L}^*(I, \theta = \varphi - Is)$ depends on the function $\tau^*(I,\varphi,s)$.
So, we need to calculate τ^* to obtain the $\mathcal{L}^*$.

From the Proposition given above, we look for τ^* such that $\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\,\tau^*, s - \tau^*) = 0$.

Different view-points for $\tau^* = \tau^*(I,\varphi,s)$

- Look for critical points of $\mathcal{L}$ on the straight line, called NHIM line $R(I,\varphi,s) = \{(I, \varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$.
- Look for intersections between $R(I,\varphi,s) = \{(I, \varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$ and a crest which is a surface of equation
$$\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\tau, s - \tau)|_{\tau=0} = 0.$$

Note that the crests are characterized by $\tau^*(I,\varphi,s) = 0$.
The crests were introduced in [Delshams-Huguet11]. A similar construction appears in [Davletshin-Treschev16].

Definition - Crests [Delshams-Huguet11]

For each I, we call *crest* $\mathcal{C}(I)$ the set of surfaces in the variables (φ, s) of equation

$$\left\langle (\omega, 1) \cdot \nabla_{(\varphi,s)} \mathcal{L}^*(I, \varphi, s) \right\rangle = 0, \tag{6}$$

where $\omega_i = \Omega_i I_i$.

which in our case can be rewritten as

$$\mu_1 \alpha(\omega_1) \sin \varphi_1 + \mu_2 \alpha(\omega_2) \sin \varphi_2 + \sin s = 0,$$

where $\mu_i = a_i / a_3$ and

$$\alpha(\omega_i) = \frac{\omega_i^2 \sinh(\frac{\pi}{2})}{\sinh(\frac{\pi \omega_i}{2})}.$$

- $\mathcal{L}^*(I, \theta)$ is nothing else but $\mathcal{L}$ evaluated on the crest $\mathcal{C}(I)$.
- $\theta = \varphi - Is$ is constant on the NHIM line $R(I, \varphi, s)$

Understanding the behavior of the crests

$\Downarrow$

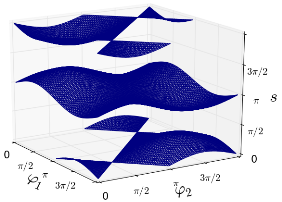
Understanding the behavior of the Reduced Poincaré function

$\Downarrow$

Understanding the Scattering map

- For $|\mu\alpha(I)| < 1$, there are two crests $\mathcal{C}_{\mathrm{M,m}}(I)$ parameterized by:

$$\begin{aligned} s = \xi_M(I,\varphi) &= -\arcsin(\mu_1\alpha(\omega_1)\sin\varphi_1 + \mu_2\alpha(\omega_2)\sin\varphi_2) && \mod 2\pi \quad (7)\\ \xi_m(I,\varphi) &= \arcsin(\mu_1\alpha(\omega_1)\sin\varphi_1 + \mu_2\alpha(\omega_2)\sin\varphi_2) + \pi && \mod 2\pi \end{aligned}$$



They are "horizontal" crests

For $0 < |\mu_1| + |\mu_2| < 0.625$:

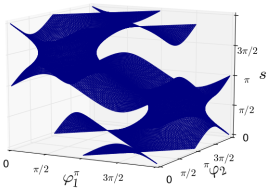
- For each I, the NHIM line $R(I, \varphi, s)$ and the crest $\mathcal{C}_{\mathrm{M,m}}(I)$ has only one intersection point.
- The scattering map S_{M} associated to the intersections between $\mathcal{C}_{\mathrm{M}}(I)$ and $R(I, \varphi, s)$ is well defined for any $\varphi \in \mathbb{T}$. Analogously for S_{m}, changing M to m. In the variables $(I, \theta = \varphi - Is)$, both scattering maps $\mathcal{S}_{\mathrm{M}}$, $\mathcal{S}_{\mathrm{m}}$ are globally well defined.

For $0.625 < |\mu_1| + |\mu_2| < 0.97$:

- There are tangencies between $\mathcal{C}_{\mathrm{M,m}}(I, \varphi)$ and $R(I, \varphi, s)$. For some value of (I, φ, s), there are 3 points in $R(I, \varphi, s) \cap \mathcal{C}_{\mathrm{M,m}}(I)$.
- This implies that there are 3 scattering maps associated to each crest with different domains.(Multiple Scattering maps)

For $|\mu_1|, |\mu_2| < 0.97$:

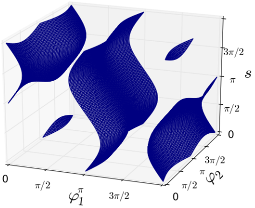
- The crests $\mathcal{C}(I)$ are horizontal or unseparated.
- For some value of I there are NHIM lines which are tangent to the crests. Again, we have multiple scattering maps.



"Unseparated" crests

For $0.97 < |\mu_1|$ or $0.97 < |\mu_2|$

- The crests $\mathcal{C}(I)$ can be horizontal, vertical or unseparated
- For some value of I there are NHIM lines which are tangent to the crests.



Example of "vertical" crests

Theorem (Arnold diffusion for a two-parameter family)

Consider the Hamiltonian (4)+(5). Assume $a_1 a_2 a_3 \neq 0$ *and* $|a_1/a_3| + |a_2/a_3| < 0.625$. *Then, for every* $\delta < 1$ *and* $R > 0$ *there exists* $\varepsilon_0 > 0$ *such that for every* $0 < |\varepsilon| < \varepsilon_0$, *given* $|I_\pm| \leq R$, *there exists an orbit* $\tilde{x}(t)$ *and* $T > 0$, *such that*

$$|I(\tilde{x}(0)) - I_-| \leq \delta \quad \text{and} \quad |I(\tilde{x}(T)) - I_+| \leq \delta.$$

Remark

Actually, we can prove that given any two actions $I_\pm$ *and any path* $\gamma(s)$ *joining them in the actions space, there exists an orbit* $\tilde{x}(t)$ *such that* $I(\tilde{x}(t))$ *is* δ*-close to* $\gamma(\Psi(t))$ *for some parameterization* Ψ.

We define a Highway as an invariant set $\mathcal{H} = \{(I, \Theta(I))\}$ of the Hamiltonian given by the reduced Poincaré function $\mathcal{L}^*(I, \theta)$ which is contained in the level energy $\mathcal{L}^*(I, \theta) = A_3$. It is therefore a Lagrangian manifold, there exists a function $F(I)$ such that $\Theta(I) = \nabla F(I)$. Therefore,

$$\frac{\partial \Theta_1}{\partial I_2} = \frac{\partial \Theta_2}{\partial I_1}, \text{ i.e., } \frac{\partial^2 F}{\partial I_2 \partial I_1} = \frac{\partial^2 F}{\partial I_1 \partial I_2}.$$

Proposition

Consider the Hamiltonian (4)+(5). *Assume* $a_1 a_2 a_3 \neq 0$ *and* $|a_1/a_3| + |a_2/a_3| < 0.625$. *For* I_1 *and* I_2 *close to infinity, the function* F *takes the asymptotic form*

$$F(I) = \frac{3\pi}{2}(I_1 + I_2) - \sum_{i=1,2} \frac{2a_i \sinh(\pi/2)}{\pi^4 \Omega_i} \left(\pi^3 \omega_i^3 + 6\pi^2 \omega_i^2 + 24\pi\omega_i + 48\right) e^{-\pi\omega_i/2}$$
$$+\mathcal{O}(\omega_1^2 \omega_2^2 e^{\pi(\omega_1+\omega_2)/2}),$$

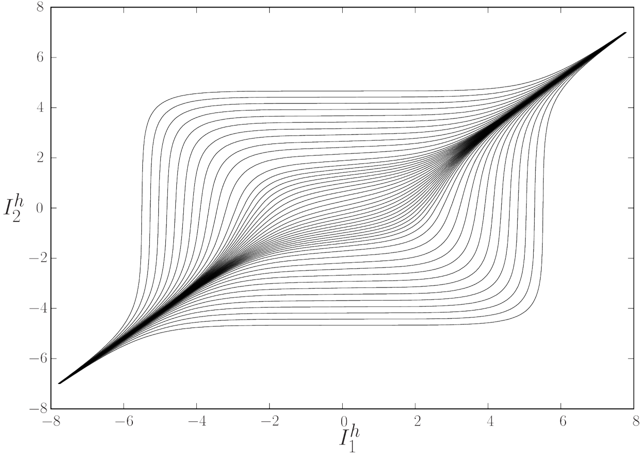


Figure: Dynamics inside the highway. Parameter values are $a_1 = 0.3$, $a_2 = 0.1$, $a_3 = 1$ and $\Omega_1 = \Omega_2 = 1$.

Proposition

Assume $a_1 a_2 a_3 \neq 0$ *and* $|a_1/a_3| + |a_2/a_3| < 0.625$ *in Hamiltonian* (4)+(5). *Let* $(I^h, \Theta(I^h))$ *a Highway. For* I_2, $I_1 \gg 1$, *we have*

$$I_2^h = \frac{\Omega_1}{\Omega_2} I_1^h + \frac{2}{\pi\Omega_2} \log\left(\frac{\Omega_2 a_2}{\Omega_1 a_1}\right),$$

and for I_2, $I_1 \ll -1$,

$$I_2^h = \frac{\Omega_1}{\Omega_2} I_1^h + \frac{2}{\pi\Omega_2} \log\left(\frac{\Omega_1 a_1}{\Omega_2 a_2}\right),$$

Theorem

The time of diffusion T_d close to a highway of Hamiltonian (4)+(5) *with* $|a_1/a_3| + |a_2/a_3| < 0.625$ *between* I_1^0 *and* I_1^f *satisfies the following asymptotic expression*

$$T_d = \frac{T_s}{\varepsilon}\left[2\log\left(\frac{C}{\varepsilon}\right) + \mathcal{O}(\varepsilon^b)\right], \ \textit{for } \varepsilon \to 0, \ \textit{where } 0 < b < 1, \qquad (8)$$

with

$$T_s = \frac{1}{2\pi a_1 \Omega_1}\int_{\omega_0}^{\omega_f} \frac{-\sinh(\pi\omega_1/2)d\omega_1}{\omega_1 \sin(\theta_1 - \omega_1\tau^*)},$$

where $\omega_0 = \Omega_1 I_1^0$ *and* $\omega_f = \Omega_1 I_f$*, and*

$$C = 16\left(|a_1| + |a_3\mu_1|\frac{2\sinh(\pi/2)\,|\mu_1|}{\pi\left[1 - 1.466(|\mu_1| + |\mu_2|)\right]} max_{I_1\in[I_1^0, I_1^f]}|\omega_1 - \alpha(I_1)| + |a_3\mu_2|\frac{2\sinh(\pi/2)\,|\mu_1|}{\pi\left[1 - 1.466(|\mu_1| + |\mu_2|)\right]} max_{[I_2(I_1^0), I_2(I_1^f)]}|\omega_2 - \alpha(I_2)|\right).$$

Grazie mille.

Thank you very much.

Moltes gràcies.

Tack så mycket.

Muchas gracias.

Muito obrigado.

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