

Global instability for 'complete' families of perturbations with two or three independent harmonics

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Consider a pendulum and a rotor plus a time periodic perturbation depending on two harmonics in the variables (φ, s) :

$$H_\varepsilon(p, q, I, \varphi, s) = \pm \left(\frac{p^2}{2} + \cos q - 1 \right) + \frac{I^2}{2} + \varepsilon h(q, \varphi, s) \quad (1)$$

$$h(q, \varphi, s) = f(q)g(\varphi, s), \quad (2)$$

$$f(q) = \cos q, \quad g(\varphi, s) = a_1 \cos(k_1\varphi + l_1s) + a_2 \cos(k_2\varphi + l_2s),$$

with $k_1, k_2, l_1, l_2 \in \mathbb{Z}$.

Theorem

Assume that $a_1 a_2 \neq 0$ and $\begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix} \neq 0$ in (1)-(2). Then, for any $I^ > 0$, there exists $\varepsilon^* = \varepsilon^*(I^*, a_1, a_2) > 0$ such that for any ε , $0 < \varepsilon < \varepsilon^*$, there exists a trajectory $(p(t), q(t), I(t), \varphi(t))$ such that for some $T > 0$*

$$I(0) \leq -I^* < I^* \leq I(T).$$

- To review the construction of scattering maps initiated in [Delshams-Llave-Seara00], designed to detect **global instability**.
- To compute **explicitly** several scattering maps to prove global instability for the action I for any $\varepsilon > 0$ small enough.
- To estimate the time of diffusion in some cases (at least for $k_1 = l_2 = 1$ and $l_1 = k_2 = 0$).
- To play with the parameter $\mu = a_1/a_2$ to prove global instability for **any value** of $\mu \neq 0, \infty$.
- To describe bifurcations of the scattering maps.

It is easy to check that if

$$\Delta := k_1 l_2 - k_2 l_1 = 0 \quad \text{or} \quad a_1 = 0 \quad \text{or} \quad a_2 = 0$$

there is no global instability for the variable l .

If $\Delta a_1 a_2 \neq 0$, after some rational linear changes in the angles, we only need to study two cases:

- The first (and easier) case [\[Delshams-S17\]](#)

$$g(\varphi, s) = a_1 \cos \varphi + a_2 \cos s$$

- The second case [\[Delshams-S18\]](#)

$$g(\varphi, \sigma) = a_1 \cos \varphi + a_2 \cos \sigma,$$

where $\sigma = \varphi - s$.

We deal with an a priori unstable Hamiltonian [Chierchia-Gallavotti94].

In the unperturbed case $\varepsilon = 0$, the Hamiltonian H_0 is integrable formed by the standard pendulum plus a rotor

$$H_0(p, q, I, \varphi, s) = \pm \left(\frac{p^2}{2} + \cos q - 1 \right) + \frac{I^2}{2}.$$

$$I \text{ is constant: } \Delta I := I(T) - I(0) \equiv 0.$$

For any $0 < \varepsilon \ll 1$, there is a finite drift in the action of the rotor I : $\Delta I = \mathcal{O}(1)$, so we have global instability.

In short, this is also frequently called Arnold diffusion.

Basically, we ensure the Arnold diffusion performing the following scheme:

- To construct iterates under several **Scattering maps** and the **Inner map**, giving rise to diffusing **pseudo-orbits**.
- To use previous results about Shadowing [[Fontich-Martín00](#)], [[Gidea-Llave-Seara14](#)] for ensuring the existence of real orbits close to the pseudo-orbits.

We have two important dynamics associated to the system: the **inner** and the **outer** dynamics on a large invariant object $\tilde{\Lambda}$.

$$\tilde{\Lambda} = \{(0, 0, I, \varphi, s); I \in [-I^*, I^*], (\varphi, s) \in \mathbb{T}^2\}.$$

is a 3D *Normally Hyperbolic Invariant Manifold* (NHIM) with associated 4D stable $W_\varepsilon^s(\tilde{\Lambda})$ and unstable $W_\varepsilon^u(\tilde{\Lambda})$ invariant manifolds.

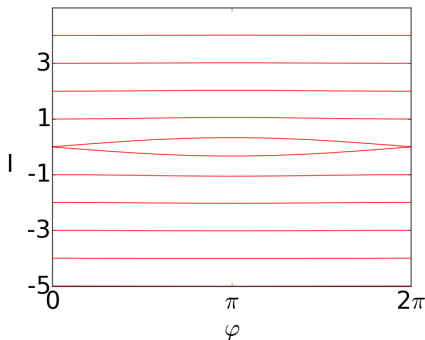
- The *inner dynamics* is the dynamics restricted to $\tilde{\Lambda}$. (**Inner map**)
- The *outer dynamics* is the dynamics along the invariant manifolds to $\tilde{\Lambda}$. (**Scattering map**)

Remark: Due to the form of the perturbation, $\tilde{\Lambda} = \tilde{\Lambda}_\varepsilon$.

For the first case $g(\varphi, s) = a_1 \cos \varphi + a_2 \cos s$, the inner dynamics is described by the Hamiltonian systems with the Hamiltonian

$$K(I, \varphi, s) = \frac{I^2}{2} + \varepsilon (a_1 \cos \varphi + \cancel{a_2 \cos s}).$$

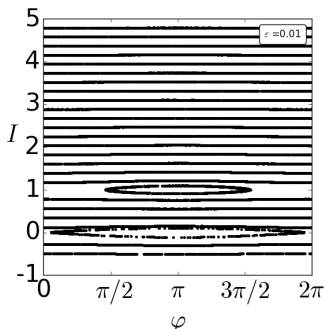
In this case the inner dynamics is integrable.



For $g(\varphi, \sigma)$, the inner dynamics is described by the Hamiltonian

$$K(I, \varphi, \sigma) = \frac{I^2}{2} + \varepsilon (a_1 \cos \varphi + a_2 \cos \sigma),$$

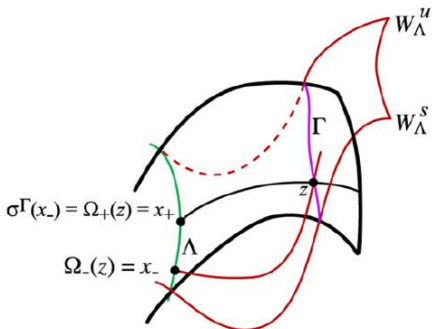
where $\sigma = \varphi - s$. The system associated to this Hamiltonian is not integrable and two resonances arise in $I = 0$ and $I = 1$.



Let $\tilde{\Lambda}$ be a NHIM with invariant manifolds intersecting transversally along a homoclinic manifold Γ . A scattering map is a map S defined by $S(\tilde{x}_-) = \tilde{x}_+$ if there exists $\tilde{z} \in \Gamma$ satisfying

$$|\phi_t^\varepsilon(\tilde{z}) - \phi_t^\varepsilon(\tilde{x}_\mp)| \rightarrow 0 \text{ as } t \rightarrow \mp\infty$$

that is, $W_\varepsilon^u(\tilde{x}_-)$ intersects transversally $W_\varepsilon^s(\tilde{x}_+)$ in \tilde{z} .



S is an exact symplectic map [Delshams-Llave-Seara08] and takes the form:

$$S_\varepsilon(I, \varphi, s) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I, \theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I, \theta) + \mathcal{O}(\varepsilon^2), s \right),$$

where $\theta = \varphi - Is$ and $\mathcal{L}^*(I, \theta)$ is the **Reduced Poincaré function**, or more simply in the variables (I, θ) :

$$S_\varepsilon(I, \theta) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I, \theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I, \theta) + \mathcal{O}(\varepsilon^2) \right),$$

- The variable s remains **fixed** under S_ε : it plays the role of a parameter
- Up to **first order** in ε , S_ε is the **$-\varepsilon$ -time flow** of the Hamiltonian $\mathcal{L}^*(I, \theta)$
- The scattering map jumps $\mathcal{O}(\varepsilon)$ distances along the **level curves** of $\mathcal{L}^*(I, \theta)$

To get a scattering map we search for homoclinic orbits to $\tilde{\Lambda}_\varepsilon$

Proposition

Given $(I, \varphi, s) \in [-I^*, I^*] \times \mathbb{T}^2$, assume that the real function

$$\tau \in \mathbb{R} \mapsto \mathcal{L}(I, \varphi - I\tau, s - \tau) \in \mathbb{R}$$

has a non degenerate critical point $\tau^* = \tau(I, \varphi, s)$, where

$$\mathcal{L}(I, \varphi, s) = \int_{-\infty}^{+\infty} (\cos q_0(\sigma) - \cos 0) g(\varphi + I\sigma, s + \sigma; 0) d\sigma.$$

Then, for $0 < |\varepsilon|$ small enough, there exists a transversal homoclinic point \tilde{z} to $\tilde{\Lambda}_\varepsilon$, which is ε -close to the point $\tilde{z}^*(I, \varphi, s) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) \in W^0(\tilde{\Lambda})$:

$$\tilde{z} = \tilde{z}(I, \varphi, s) = (p_0(\tau^*) + O(\varepsilon), q_0(\tau^*) + O(\varepsilon), I, \varphi, s) \in W^u(\tilde{\Lambda}_\varepsilon) \cap W^s(\tilde{\Lambda}_\varepsilon).$$

In our model $q_0(t) = 4 \arctan e^t$, $p_0(t) = 2/\cosh t$ is the **separatrix** for positive p of the standard pendulum $P(q, p) = p^2/2 + \cos q - 1$.

- For $g(\varphi, s) = a_1 \cos \varphi + a_2 \cos s$, the Melnikov potential becomes

$$\mathcal{L}(I, \varphi, s) = A_1(I) \cos \varphi + A_2 \cos s,$$

where $A_1(I) = \frac{2\pi I a_1}{\sinh\left(\frac{I\pi}{2}\right)}$ and $A_2 = \frac{2\pi a_2}{\sinh\left(\frac{\pi}{2}\right)}$.

- For $g(\varphi, \sigma) = a_1 \cos \varphi + a_2 \cos \sigma$ ($\sigma = \varphi - s$), the Melnikov potential becomes

$$\mathcal{L}(I, \varphi, \sigma) = A_1(I) \cos \varphi + A_2(I) \cos \sigma,$$

where $A_1(I)$ is as before but now $A_2(I) = \frac{2(I-1)\pi a_2}{\sinh\left(\frac{(I-1)\pi}{2}\right)}$.

The Melnikov potentials are similar in both cases.

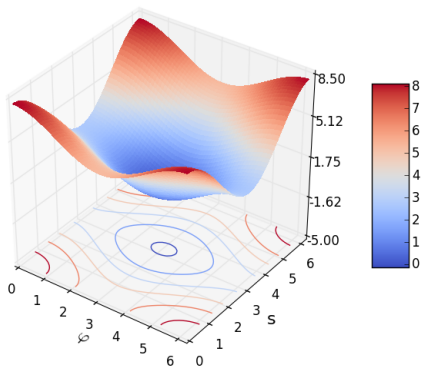


Figure: The Melnikov Potential, $\mu = a_1/a_2 = 0.6$, $l = 1$, $g(\varphi, s)$.

Finally, the function $\mathcal{L}^*(I, \theta)$ can be defined:

Definition

The **Reduced Poincaré function** is

$$\mathcal{L}^*(I, \theta) = \mathcal{L}(I, \varphi - I\tau^*(I, \varphi, s), s - \tau^*(I, \varphi, s)),$$

where $\theta = \varphi - Is$.

Therefore the definition of $\mathcal{L}^*(I, \theta)$ depends on the function $\tau^*(I, \varphi, s)$.

From the Proposition given above, we look for τ^* such that

$$\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\tau^*, s - \tau^*) = 0.$$

Different view-points for $\tau^* = \tau^*(I, \varphi, s)$

- Look for critical points of \mathcal{L} on the straight line, called **NHIM line**

$$R(I, \varphi, s) = \{(I, \varphi - I\tau, s - \tau), \tau \in \mathbb{R}\}.$$

- Look for intersections between

$R(I, \varphi, s) = \{(I, \varphi - I\tau, s - \tau), \tau \in \mathbb{R}\}$ and a **crest** which is a curve of equation

$$\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\tau, s - \tau)|_{\tau=0} = 0.$$

Note that the crests are characterized by $\tau^*(I, \varphi, s) = 0$.

The crests were introduced in [Delshams-Huguet11]. A similar construction appears in [Davletshin-Treschev16].

Definition - Crests [Delshams-Huguet11]

For each I , we call *crest* $\mathcal{C}(I)$ the set of curves in the variables (φ, s) of equation

$$I \frac{\partial \mathcal{L}}{\partial \varphi}(I, \varphi, s) + \frac{\partial \mathcal{L}}{\partial s}(I, \varphi, s) = 0. \quad (3)$$

which in our case can be rewritten as

$$g(\varphi, s): \mu \alpha(I) \sin \varphi + \sin s = 0, \quad \text{with } \alpha(I) = \frac{I^2 \sinh(\frac{\pi}{2})}{\sinh(\frac{\pi I}{2})}, \quad \mu = \frac{a_1}{a_2}.$$

$$g(\varphi, \sigma = \varphi - s): \mu \alpha(I) \sin \varphi + \sin \sigma = 0, \quad \text{with } \alpha(I) = \frac{I^2 \sinh(\frac{(I-1)\pi}{2})}{(I-1)^2 \sinh(\frac{\pi I}{2})}, \quad \mu = \frac{a_1}{a_2}.$$

- For any I , the critical points of the Melnikov potential $\mathcal{L}(I, \cdot, \cdot)$ $((0, 0)$, $(0, \pi)$, $(\pi, 0)$ and (π, π)): one maximum, one minimum point and two saddle points) always belong to the crest $\mathcal{C}(I)$.
- $\mathcal{L}^*(I, \theta)$ is nothing else but \mathcal{L} evaluated on the crest $\mathcal{C}(I)$.
- $\theta = \varphi - Is$ is constant on the NHIM line $R(I, \varphi, s)$

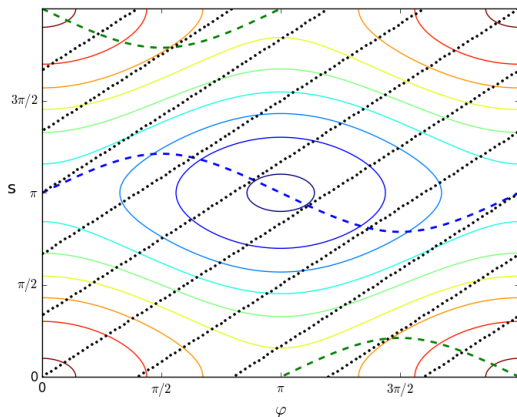


Figure: Level curves of \mathcal{L} for $\mu = a_1/a_2 = 0.5$, $l = 1.2$ and $g(\varphi, s)$.

Understanding the behavior of the crests



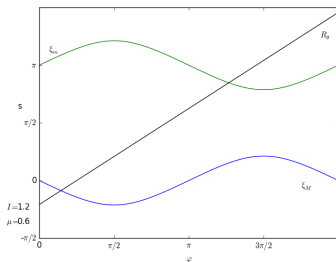
Understanding the behavior of the Reduced Poincaré function



Understanding the Scattering map

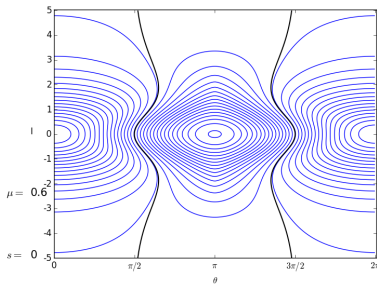
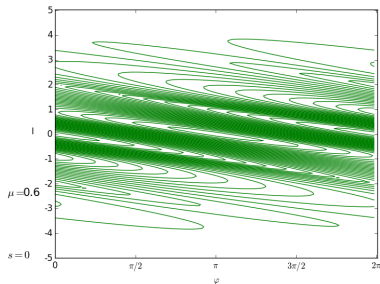
- $|\mu\alpha(I)| < 1$ for all values of I , there are two crests $\mathcal{C}_{M,m}(I)$ parameterized by:

$$\begin{aligned} s = \xi_M(I, \varphi) &= -\arcsin(\mu\alpha(I) \sin \varphi) \quad \text{mod } 2\pi \\ \xi_m(I, \varphi) &= \arcsin(\mu\alpha(I) \sin \varphi) + \pi \quad \text{mod } 2\pi \end{aligned} \quad (4)$$

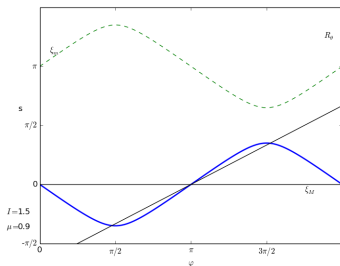


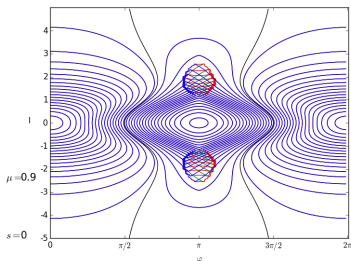
They are “horizontal” crests

- For each I , the NHIM line $R(I, \varphi, s)$ and the crest $\mathcal{C}_{M,m}(I)$ has only one intersection point.
- The scattering map S_M associated to the intersections between $\mathcal{C}_M(I)$ and $R(I, \varphi, s)$ is well defined for any $\varphi \in \mathbb{T}$. Analogously for S_m , changing M to m . In the variables $(I, \theta = \varphi - Is)$, both scattering maps S_M, S_m are globally well defined.

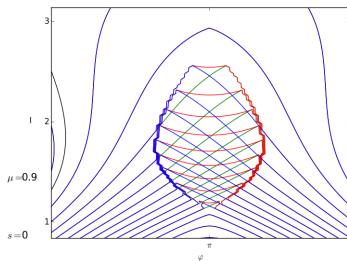
(a) Level curves of $\mathcal{L}_M^*(I, \theta)$ (b) Level curves of $\mathcal{L}_m^*(I, \theta)$

- There are **tangencies** between $\mathcal{C}_{M,m}(I, \varphi)$ and $R(I, \varphi, s)$. For some value of (I, φ, s) , there are **3** points in $R(I, \varphi, s) \cap \mathcal{C}_{M,m}(I)$.
- This implies that there are **3** scattering maps associated to each crest with different domains. (**Multiple Scattering maps**)





(c) The three types of level curves.

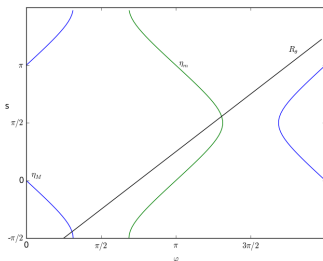


(d) Zoom where the scattering maps are different

Figure: Level curves of $\mathcal{L}_M^*(I, \theta)$, $\mathcal{L}_M^{*(1)}(I, \theta)$ and $\mathcal{L}_M^{*(2)}(I, \theta)$

- For some values of I , $|\mu\alpha(I)| > 1$, the two crests $\mathcal{C}_{M,m}$ are parameterized by:

$$\begin{aligned}\varphi = \eta_M(I, s) &= -\arcsin(\mu\alpha(I) \sin s) \quad \text{mod } 2\pi \\ \eta_m(I, s) &= \arcsin(\mu\alpha(I) \sin s) + \pi \quad \text{mod } 2\pi\end{aligned}\quad (5)$$



They are “vertical” crests

For the values of I for which horizontal crests become vertical, it is not always possible to prolong in a continuous way the scattering maps, so the domain of the scattering map has to be restricted.

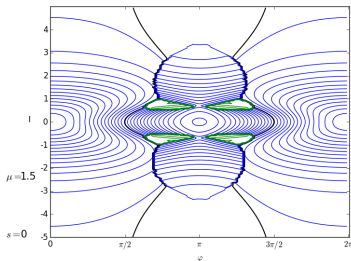


Figure: The level curves of $\mathcal{L}_M^*(I, \theta)$, $\mu = 1.5$.

In green, the region where the scattering map S_M is not defined.

Definition: Highways

Highways are the level curves of \mathcal{L}^* such that

$$\mathcal{L}^*(I, \theta) = \frac{2\pi a_1}{\sinh(\pi/2)}.$$

- The highways are “vertical” in the variables (φ, s)
- We always have a pair of highways. One goes up, the other goes down (this depends on the sign of $\mu = a_1/a_2$)
- The highways give rise to fast diffusing pseudo-orbits

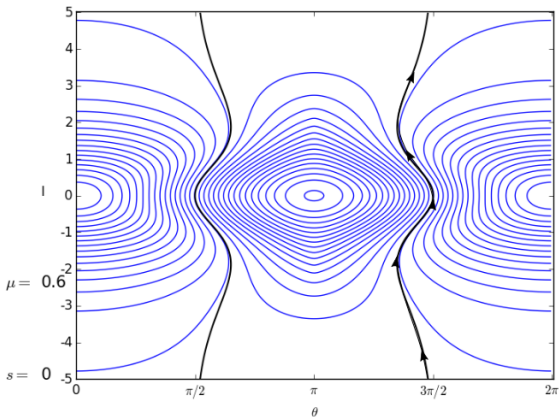


Figure: The scattering map jumps $\mathcal{O}(\varepsilon)$ distances along the level curves of $\mathcal{L}^*(I, \theta)$

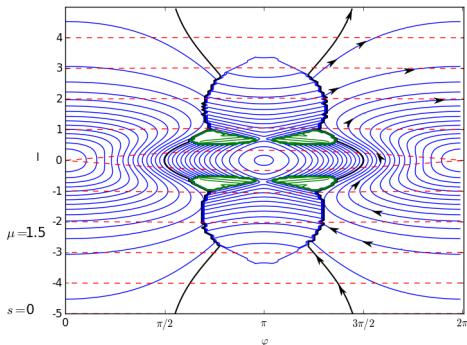


Figure: In red: Inner map, blue: Scattering map, black: Highways, $\mu = 1.5$.

An estimate of the total time of diffusion between $-I^*$ and I^* , along the highway, is

$$T_d = \frac{T_s}{\varepsilon} \left[2 \log \left(\frac{C}{\varepsilon} \right) + \mathcal{O}(\varepsilon^b) \right], \text{ for } \varepsilon \rightarrow 0, \text{ where } 0 < b < 1,$$

with

$$T_s = T_s(I^*, a_1, a_2) = \int_0^{I^*} \frac{-\sinh(\pi l/2)}{\pi a_1 l \sin \psi_h(l)} dl,$$

where $\psi_h = \theta - I\tau^*(l, \theta)$ is the parameterization of the highway $\mathcal{L}^*(l, \psi_h) = A_2$, and

$$C = C(I^*, a_1, a_2) = 16 |a_1| \left(1 + \frac{1.465}{\sqrt{1 - \mu^2 A^2}} \right)$$

where $A = \max_{l \in [0, I^*]} \alpha(l)$, with $\alpha(l) = \frac{\sinh(\frac{\pi}{2}) l^2}{\sinh(\frac{\pi l}{2})}$ and $\mu = a_1/a_2$.

Note: This estimate agrees with the upper bounds given in [Bessi-Chierchia-Valdinoci01] and quantifies the general optimal diffusion estimate $\mathcal{O}\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right)$ of [Berti-Biasco-Bolle03] and [Treschev04].

In the second case:

- For $|\mu\alpha(I)| < 1$, there are two crests $\mathcal{C}_{M,m}(I)$ parameterized by $\sigma = \xi_M(I, \varphi)$ and $\xi_m(I, \varphi)$. For $|\mu\alpha(I)| > 1$, $\mathcal{C}_{M,m}(I)$ parameterized by $\varphi = \eta_M(I, \sigma)$ and $\eta_m(I, \sigma)$. **The crests lie on the plane (φ, σ)**
- There are no *Highways*.
- For any value of $\mu = a_1/a_2$ is possible to find I_h and I_v such that for $I = I_h$ the crests are horizontal and for $I = I_v$ the crests are vertical.
- For any value of μ there exists I such that the crests and some NHIM line are tangent. **There are always multiple scattering maps**

From the definitions of $R(I, \varphi, s)$ and $\mathcal{C}(I)$, we have

$$R(I, \varphi, s) \cap \mathcal{C}(I) = \{(I, \varphi - I\tau^*(I, \varphi, s), s - \tau^*(I, \varphi, s))\}.$$

Introducing

$$\tau^*(I, \theta) := \tau^*(I, \varphi - Is, 0), \quad \text{with } \theta = \varphi - Is = (1 - I)\varphi + I\sigma,$$

one can see that on the plane $(\varphi, \sigma = \varphi - s)$, the NHIM lines take the form

$$R_I(\varphi, \sigma) = \{(\varphi - I\tau, \sigma - (I - 1)\tau), \tau \in \mathbb{R}\}$$

and that

$$R_I(\varphi, \sigma) \cap \mathcal{C}(I) = \{(\theta - I\tau^*(I, \theta), \theta - (I - 1)\tau^*(I, \theta))\}.$$

Therefore, the function $\tau^*(I, \theta)$ is the time spent to go from a point (θ, θ) in the diagonal $\sigma = \varphi$ up to $\mathcal{C}(I)$ with a velocity vector $\mathbf{v} = -(I, I - 1)$.

The choice of the concrete curve of the crest and therefore of $\tau^*(I, \theta)$ is very important and useful.

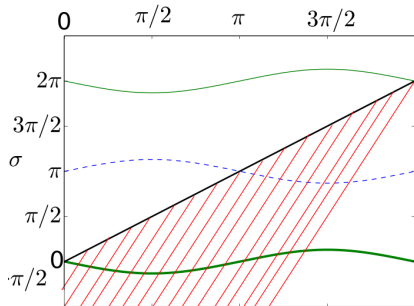


Figure: Going down along NHIM lines

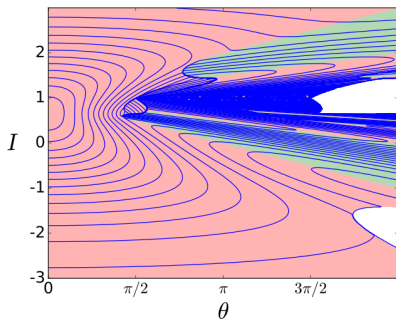


Figure: The “lower” crest

Green zones: I increases under the scattering map.

Red zones: I decreases under the scattering map.

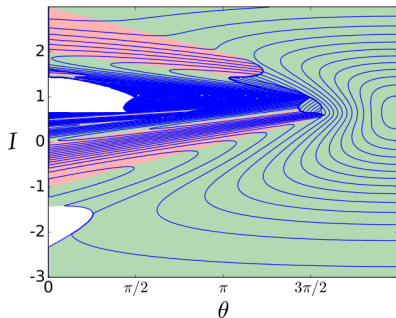
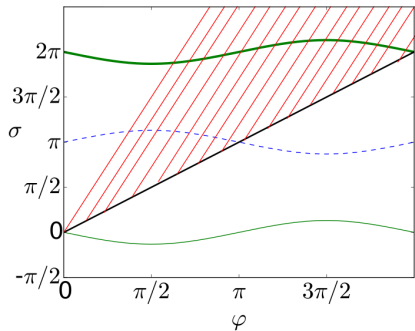


Figure: Going up along NHIM lines

Figure: The “upper” crest

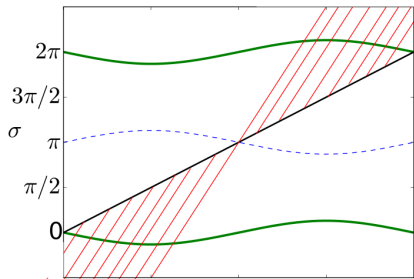


Figure: Minimal time

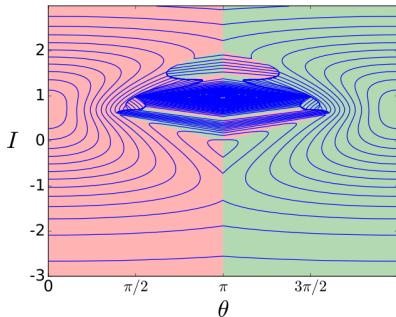


Figure: Minimal $|\tau^*|$ between “lower” and “upper” crest

In this picture we show a combination of 3 scattering maps.

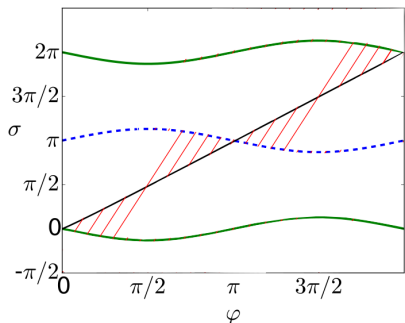


Figure: First intersection

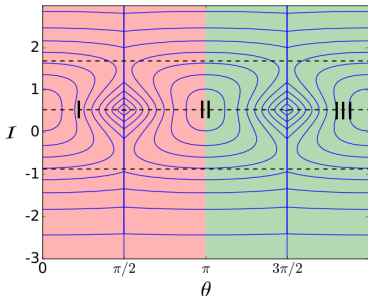


Figure: Minimal $|\tau^*|$ between $C_M(I)$ and $C_m(I)$

Consider a pendulum and **two** rotors plus a time periodic perturbation depending on three harmonics in the angles $(\varphi_1, \varphi_2, \varphi_3 = s)$

$$H_\varepsilon(p, q, I, \varphi, s) = \pm \left(\frac{p^2}{2} + \cos q - 1 \right) + h(I) + \varepsilon f(q) g(\varphi, s), \quad (6)$$

where $I = (I_1, I_2)$, $\varphi = (\varphi_1, \varphi_2)$, $f(q) = \cos q$, $h(I) = \Omega_1 I_1^2/2 + \Omega_2 I_2^2/2$ and

$$g(\varphi, s) = a_1 \cos \varphi_1 + a_2 \cos \varphi_2 + a_3 \cos s. \quad (7)$$

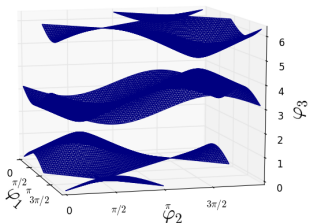
Theorem (Arnold diffusion for a two-parameter family)

Assume $a_1 a_2 a_3 \neq 0$ and $|a_1/a_3| + |a_2/a_3| < 0.625$ in Hamiltonian (6)+(7). Then, for any two actions I_\pm and any δ there exists $\varepsilon_0 > 0$ such that for every $0 < |\varepsilon| < \varepsilon_0$ there exists an orbit $\tilde{x}(t)$ and $T > 0$ such that

$$|I(\tilde{x}(0)) - I_-| \leq \delta \quad \text{and} \quad |I(\tilde{x}(T)) - I_+| \leq \delta$$

For $|a_1/a_3| + |a_2/a_3| < 0.625$ there are two horizontal crests $\mathcal{C}_{M,m}(I)$, and both scattering maps $\mathcal{S}_M, \mathcal{S}_m$ are globally well defined.

Figure: Horizontal crests: $a_1/a_3 = a_2/a_3 = 0.48$, $\Omega_1 I_1 = \Omega_2 I_2 = 1.219$.



Diffusing orbits are found by shadowing orbits of both scattering maps and the inner dynamics.

Remark

Actually, we can prove that given any two actions I_{\pm} and any path $\gamma(s)$ joining them in the actions space, there exists an orbit $\tilde{x}(t)$ such that $I(\tilde{x}(t))$ is δ -close to $\gamma(\Psi(t))$ for some parameterization Ψ .

Highways is an invariant set $\mathcal{H} = \{(I, \Theta(I))\}$ of the Hamiltonian given by the reduced Poincaré function $\mathcal{L}^*(I, \theta)$ which is contained in the level energy $\mathcal{L}^*(I, \theta) = A_3$. Then, $\Theta(I)$ is a Lagrangian manifold, there exists a function $F(I)$ such that $\Theta(I) = \nabla F(I)$.

Proposition

Consider the Hamiltonian (6)+(7). Assume $a_1 a_2 a_3 \neq 0$ and $|a_1/a_3| + |a_2/a_3| < 0.625$. For I_1 and I_2 close to infinity, the function F takes the asymptotic form

$$F(I) = \frac{3\pi}{2} (I_1 + I_2) - \sum_{i=1,2} \frac{2a_i \sinh(\pi/2)}{\pi^4 \Omega_i} (\pi^3 \omega_i^3 + 6\pi^2 \omega_i^2 + 24\pi \omega_i + 48) e^{-\pi \omega_i/2} + \mathcal{O}(\omega_1^2 \omega_2^2 e^{\pi(\omega_1 + \omega_2)/2}),$$

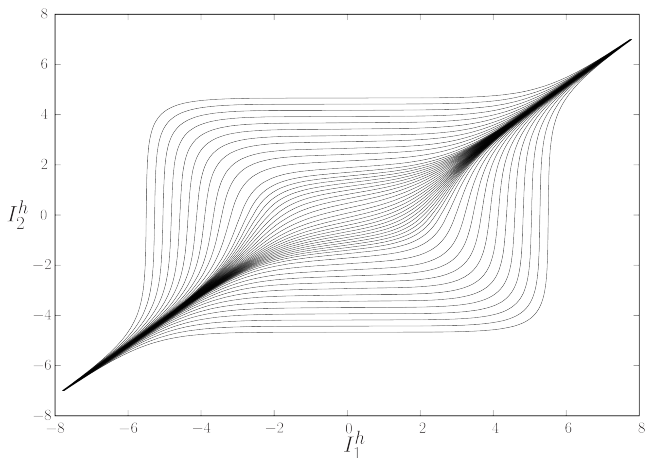


Figure: Dynamics inside the highway. Parameter values are $a_1 = 0.3$, $a_2 = 0.1$, $a_3 = 1$ and $\Omega_1 = \Omega_2 = 1$.

Theorem

The time of diffusion T_d close to a highway of Hamiltonian (6)+(7) with $|a_1/a_3| + |a_2/a_3| < 0.625$ between I_1^0 and I_1^f satisfies the following asymptotic expression

$$T_d = \frac{T_s}{\varepsilon} \left[2 \log \left(\frac{C}{\varepsilon} \right) + \mathcal{O}(\varepsilon^b) \right], \text{ for } \varepsilon \rightarrow 0, \text{ where } 0 < b < 1, \quad (8)$$

with

$$T_s = \frac{1}{2\pi a_1 \Omega_1} \int_{\omega_0}^{\omega_f} \frac{-\sinh(\pi\omega_1/2) d\omega_1}{\omega_1 \sin(\theta_1 - \omega_1 \tau^*)},$$

where $\omega_0 = \Omega_1 I_1^0$ and $\omega_f = \Omega_1 I_1^f$, and

$$C = 16 \left(|a_1| + |a_3 \mu_1| \frac{2 \sinh(\pi/2) |\mu_1|}{\pi [1 - 1.466(|\mu_1| + |\mu_2|)]} \max_{I_1 \in [I_1^0, I_1^f]} |\omega_1 - \alpha(I_1)| \right. \\ \left. + |a_3 \mu_2| \frac{2 \sinh(\pi/2) |\mu_1|}{\pi [1 - 1.466(|\mu_1| + |\mu_2|)]} \max_{[I_2(I_1^0), I_2(I_1^f)]} |\omega_2 - \alpha(I_2)| \right).$$

Thank you very much.

Tack så mycket.

Moltes gràcies.

Muchas gracias.

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